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SEMESTER – III

CORE PAPER –V

REAL ANALYSIS - I (90 Hours) (SMMA31)

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Objectives:
-To lay a god foundation of classical analysis
-To study the behaviour of sequences and series

Unit I **Real number system :**
The field of axioms, the order axioms, the rational numbers, the irrational numbers, upper bounds, maximum element, least upper bound (supremum). The completeness axiom, absolute values, the triangle inequality. Cauchy – schwartz's inequality. **11L**

Unit II **Sequences :** Bounded sequences – monotonic sequences – convergent sequences – divergent and oscillating sequences – The algebra of limits. **17L**

Unit III Behaviour of monotonic sequences – Cauchy's first limit theorem – Cauchy's second limit theorem – Cesaro's theorem – subsequences - Cauchy sequence – Cauchy's general principle of convergence. **19L**

Unit IV **Series :** Infinite series – n^{th} term test – Comparison test – Kummer's test – D'Alemberls ratio test – Raabe's test - Gauss test – Root test **23L**

Unit V Alternating series – Leibnitz's test - Tests for convergence of series of arbitrary terms – Multiplication of series- Abel's Throrem-Mertens theorem-Power Series-Radius of convergence **20L**

Text Books:

- Arumugam .S and Thengapandi Issac – "sequences and series", New Gamma publishing House, Palayamkottai – 627 002.
- Tom M. Apostol – Mathematical Analysis, II Edition, Narosa Publishing House, New Delhi (unit I)

Book for Reference :

- Goldberg .R – Methods of Real Analysis, Oxford and IBH Publishing Co., New Delhi.

UNIT I

REAL NUMBER SYSTEM

Field Axioms

Axiom 1: x + y = y + x, xy = yx (Commutative law)

Axiom 2: x +(y +z) = (x+y)+z, x(yz) = (xy)z (Associative law)

Axiom 3: x(y+z) =xy + xz (Distributive law)

Axiom 4: Given any two real number x and y there exists a real number z such that

x + z = y …..(1)

z = y – x

x+ (y –x) = y

(x-x) +y = y

x –x = 0

Therefore x is a negative of x.

Axiom 5: There exists atleast one real number x ≠ 0. If x and y are two real numbers with x ≠ 0. There exists a real number z such that xz =y implies z = $\frac{y}{x}$.

x ($\frac{y}{x}$) = y

($\frac{x}{x}$) y = y

⇒ ($\frac{x}{x}$) y = 1.y

⇒ x x^{-1} =1 $\frac{y}{x}$.

⇒ x^{-1} = $\frac{1}{x}$, x ≠ 0.’’

$\frac{1}{x}$ is the inverse of x.

x^{-1} is the reciprocal of x.

Order Axioms

The existence of a relation < which establishes an ordering among the real numbers and which satisfy the following axioms.

Axiom 6: Exactly one of the relations x = y, x < y, x > y holds.

Axiom 7: If x < y,then for all z,we have x +z < y+z.

Axiom 8: If x > 0 and y > 0 then xy > 0

Axiom 9: If x > y and y > z then x > z

Rational Numbers

Q = { $\frac{a}{b}$ / a and b are integers b ≠ 0}

Example

1. If a and b are rational numbers ,then $\frac{a+b}{2}$ is also a rational number.
2. Between any two rational numbers ,there are infinitely many rational numbers.
3. The field axiom and order axioms are satisfied by Q.

Irrational Numbers

Real numbers which are not rational are called irrational numers

Theorem 1.1 : Given real numbers a and b such that a ≤ b +∈ for all ∈ >0. Then a ≤ b.

Proof: We have to prove this theorem by contradiction method. Suppose a>b.

Given a , b ∈ R, a ≤ b +∈, for all ∈ >0. Take ∈ = $\frac{a-b}{2}$.

Then b + ∈ = $\frac{a-b}{2}$+ b.

b + ∈ = $\frac{a-b+2b}{2}$

b +∈ < a.

But given b+∈ ≥ a,which is a contradiction.

Therefore our assumption is wrong.

Thus a ≤ b.

Definition :

A subset A of R is said to be **bounded above** if there exists an element α ϵ R such that $a \leq \alpha$ *for all* $a \epsilon A$.

α is called an **upper bound** of A.

Definition :

A subset A of R is said to be **bounded below** if there exists an element β ϵ R such that $a \geq \beta$ *for all* $a \epsilon A$.

β is called a **lower bound** of A.

Definition :

A is said to be **bounded** if it is both bounded above and bounded below.

Least Upper Bound and Greatest Lower Bound:

Definition :

Let A be a subset of R and $u \epsilon R$. u is called **the least upper bound or supremum** of A if i) u is an upper bound of A.

ii) $v < u$ then v is not an upper bound of A.

Definition :

Let A be a subset of R and $l \,\varepsilon\, R$. l is called **the greatest lower bound or infimum** of A if i) l is a lower bound of A.

if $m < l$ then m is not a lower bound of A.

Examples:

1. Let A = {1, 3, 5, 6}. Then glb of A = 1 and lub of A = 6
2. Let A = (0,1). Then glb of A = 0 and lub of A = 1. In this case both glb and lub do not belong to A.

Bounded Functions:

Definition:

Let $f : A \to R$ be any function. Then the range of f is a subset of R. f is said to be **bounded function** if its range is a bounded subset of R.

Remark :

f is a bounded function iff there exists a real number m such that $|f(x)| \leq m$ *for all* $x \epsilon R$.

1. f : [0,1] →R given by f(x) = x + 2 is a bounded function where as f : R ⟶ R given by $f(x) = x + 2$ is not a bounded function.
2. f :R→ R defined by f(x) =sin x is a bounded function. Since $|\sin x| \leq 1$.

Absolute Value:

Definition: For any real number x we defined the **modulus** or the **absolute value** of x denoted by |x| as follows $|x| = \begin{cases} x & if\ x > 0 \\ -x & if x \leq 0 \end{cases}$.

Clearly $|x| \geq 0\ for\ all\ x \in R.$

Triangle inequality

For arbitrary real x and y we have | x + y | ≤ | x | + | y |

Proof:

We know that - | x | ≤ x ≤ | x | ⟶ (1)

and - | y | ≤ y ≤ | y | ⟶ (2)

(1)+(2) => - [| x | + | y |] ≤ x + y ≤ | x | + | y | .

By theorem, " If a ≥ 0 , then we have the inequality | x | ≤ a iff - a ≤ x ≤ a " .

Hence, | x + y | ≤ | x | + | y |.

Cauchy-schwarz inequality

Theorem:1.1 If $a_1, \dots a_n$ and $b_1, \dots b_n$ are real numbers, then

$(\sum_{i=1}^{n} a_i b_i)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$ ....(1)

Or, equivalently

$|\sum_{i=1}^{n} a_i\, b_i| \leq \sqrt{\sum_{i=1}^{n} a_i^2}\ \sqrt{\sum_{i=1}^{n} b_i^2}$ ....(2)

We will use mathematical induction as a method for the proof. First we observe that $(a_1b_2 - a_2b_1)^2 \geq 0$

By expanding the square we get

$$(a_1b_2)^2 + (a_2b_1)^2 - 2a_1b_2a_2b_1 \geq 0$$

After rearranging it further and completing the square on the left-hand side, we get

$$a_1{}^2b_1{}^2 + 2a_1b_1a_2b_2 + a_2{}^2b_2{}^2 \leq a_1{}^2b_1{}^2 + a_1{}^2b_2{}^2 + a_2{}^2b_1{}^2 + a_2{}^2b_2{}^2$$

By taking the square roots of both sides, we reach

$$|a_1b_1 + a_2b_2| \leq \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2} \quad \text{...........(3)}$$

which proves the inequality (2) for n = 2.

Assume that inequality (2) is true for any n terms. For n + 1, we have that

$$\sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} = \sqrt{\sum_{i=1}^{n} a_i^2 + a_{n+1}^2}\sqrt{\sum_{i=1}^{n} b_i^2 + b_{n+1}^2} \quad \text{..........................(4)}$$

By comparing the right-hand side of equation (4) with the right-hand side of inequality (3)

we know that

$$\sqrt{\sum_{i=1}^{n} a_i^2 + a_{n+1}^2}\sqrt{\sum_{i=1}^{n} b_i^2 + b_{n+1}^2} \geq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} + |a_{n+1}b_{n+1}|$$

Since we assume that inequality (2) is true for n terms, we have that

$$\sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} + |a_{n+1}b_{n+1}| \geq \sum_{i=1}^{n} a_ib_i + |a_{n+1}b_{n+1}|$$

$$\geq \sum_{i=1}^{n} a_ib_i$$

which proves the C-S inequality.

Theorem:1.2

Given real numbers a and b such that $a \leq b + \varepsilon$ for every $\varepsilon > 0$. Then $a \leq b$

Proof:

Given $a \leq b + \varepsilon \ for \ every \ \varepsilon > 0$ ....(1)

Suppose $b < a$

Choose $\varepsilon = a - b/2$

Now, $b + \varepsilon = b + a - b/2$

$$= (2b + a - b)/2$$
$$= (a + b)/2 < (a + a)/2$$
$$= 2a/2 = a$$

Therefore, $b + \varepsilon < a$, which is a contradiction to (1)
Hence $a \leq b$

Theorem: 1.3

If n is positive integer which is not a perfect square, then $\sqrt{n}$ is irrational.

Proof:

Let n contains no square factor > 1

Suppose $\sqrt{n}$ is rational

Then $\sqrt{n} = a/b$, where a and b are integers having no factor in common.

implies $n = \frac{a^2}{b^2}$

$\Rightarrow b^2 n = a^2$......(1)

But b^2n is a multiple of n, so a^2 is also a multiple of n

However if a^2 is a multiple of n, a itself must be a multiple of n. (since n has no square factor >1)

$\Rightarrow a = c\,n$, where c is an integer

sub in (1)

$b^2 n = c^2n^2$

$b^2 = nc^2$

Therefore b is a multiple of n, which is a contradiction to a and b have no factor in common.

Hence $\sqrt{n}$ is irrational

If n has a square factor, then n = m^2 k, where k> 1 and k has no square factor >1.

Then $\sqrt{n} = m\sqrt{k}$

If $\sqrt{n}$ is rational, then the numbers $\sqrt{k}$ is also rational. Which is a contradiction to k is no square factor > 1. Hence n has no square factor.

Problem:

Prove that $\sqrt{2}$ is irrational.

Theorem : If e^x = 1 + x + $\frac{x^2}{2!}$ + $\frac{x^3}{3!}$ + $\frac{x^4}{4!}$ + ….+ $\frac{x^n}{n!}$ +…, then e is irrational.

Proof: Let x =1. Then e^1 = 1 + 1 + $\frac{1^2}{2!}$ + $\frac{1^3}{3!}$ + $\frac{1^4}{4!}$ +….

and let x = -1.Then e^{-1} = 1 - 1 + $\frac{1}{2!}$ - $\frac{1}{3!}$ + $\frac{1}{4!}$ - ….

$S_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ then $S_{2k-1} = \sum_{k=0}^{n} \frac{(-1)^{2k-1}}{(2k-1)!}$

$e^{-1} - S_{2k-1} = \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots.$

$0 < e^{-1} - S_{2k-1} < \frac{1}{(2k)!}$

$0 < e^{-1} - S_{2k-1} < \frac{1}{(2k-1)!2k}$

$0 < (2k-1)!\,(e^{-1} - S_{2k-1}) = \frac{1}{2k} \leq$ ½ for any integer k ≥ 1.

Since $(2k-1)!$ Is an integer, $(2k-1)!\,(e^{-1} - S_{2k-1})$ is always an integer.

$0 < (2k-1)!\,(e^{-1} - S_{2k-1}) = \frac{1}{2k} \leq$ ½

If e^{-1} is rational, $(2k-1)!\ e^{-1}$ is an integer, which would lie between 0 and ½.

Which is a contradiction.

Hence, e cannot be rational.

UNIT - II
SEQUENCES

Definition. Let f : ℕ → ℝ be a function and let f (n) = a_n. Then $a_1, a_2, \dots, a_n$ is called the sequences in ℝ determined by the function f and is denoted by (a_n).

a_n is called the n^{th} term of the sequence. The range of the function f which is a subset of ℝ, is called the range of the sequence

Examples.

a) The function f : ℕ → ℝ given by $f\ (n)\ =\ n$ determines the sequence 1, 2, 3, ..., ...,n,

b) The function f : ℕ→ ℝ given by $f\ (n)\ =\ n^2$ determines the sequence 1, 4, 9, ...,...,n^2,...

Definition:

A sequence (a_n) is said to be **bounded above** if there exists a real number k such that $a_n \leq$ k for all **n** ∈ℕ. k is called an upper bound of the sequence(a_n) .

A sequence (a_n) is said to be **bounded below** if there exists a real number k such that$a_n \geq$ k for all n. k is called a **lower bound** of the sequence (a_n) .

A sequence (a_n) is said to be a **bounded sequence** if it is both bounded above and below.

Note.

A sequence(a_n) is bounded if there exists a real number $k > 0$ such that $|a_n| < k\ for\ all n$

Examples.

1. Consider the sequence 1, 1/2 , 1/3,.... 1/n.... Here 1 is the l.u.b and 0 is the g.l.b. It is a bounded sequence.

2. The sequence 1, 2, 3, ..., n,.......... is bounded below but not bounded above. 1 is the g. l.b of the sequence.

3. The sequence–1,–2,–3,...–n,...is bounded above but not bounded below.

–1 is the l.u.b of the sequence.

4. 1, –1, 1, –1, ... is a bounded sequence. 1 is the l. u. b –1 is the g. l. b of the sequence

5. Any constant sequence is a bounded sequence. Here 1.u.b = g. l. b = the constant term of the sequence.

Monotonic sequence

Definition: A sequence (a_n) is said to be monotonic increasing if $a_n \leq a_{n+1}$ for all n. (a_n) is said to be monotonic decreasing if $a_n \geq a_{n+1}$ for all n. (a_n) is said to be strictly monotonic decreasing if $a_n < a_{n+1}$ for all n. (a_n) is said to be monotonic if it is either monotonic increasing or monotonic decreasing.

Example.

1. 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ... is a monotonic increasing sequence.
2. 1,2,3,4... ...... is a strictly monotonic increasing sequence
3. The sequence (a_n) given by 1, –1, 1, –1, 1, ... is neither monotonic increasing nor monotonic decreasing. Hence(a_n) is not a monotonic sequence.
4. $\left(\frac{2n-7}{3n+2}\right)$ is a monotonic increasing sequence.

Proof:

$$a_n - a_{n+1} = \frac{2n-7}{3n+2} - \frac{2(n+1)-7}{3(n+1)+2}$$
$$= \frac{-25}{(3n+2)(3n+5)} < 0$$

Therefore $a_n < a_{n+1}$

Hence the sequence is monotonic increasing.

5. Consider the sequence (a_n) where $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots . + \frac{1}{n!}$. Clearly (a_n) is a monotonic increasing sequence.

Note: A monotonic increasing sequence (a_n) is bounded below and q_1 is the g.l.b of the sequence.

A monotonic decreasing sequence (a_n) is bounded above and a_1 is l. u. b of the sequence.

Solved Problems:

Show that if (a_n) is a monotonic sequence then $\left(\frac{a_1+a_2+\cdots\ldots+a_n}{n}\right)$ is also a monotonic sequence.

Solution:

Let (a_n) be a monotonic increasing sequence.

Therefore $a_1 \leq a_2 \leq a_3 \leq \cdots \ldots \leq a_n \leq$ ....(1)

Let $b_n = \left(\frac{a_1+a_2+\cdots\ldots+a_n}{n}\right)$

Now, $b_{n+1} - b_n = \frac{a_1+a_2+\cdots\ldots+a_{n+1}}{n+1} - \frac{a_1+a_2+\cdots\ldots+a_n}{n}$

$\geq \frac{na_{n+1} - (a_1+a_2+\cdots\ldots+a_n)}{n(n+1)}$

$= \frac{na_{n+1} - (a_n+a_n+\cdots\ldots+a_n)}{n(n+1)}$ by (1)

$= \frac{n(a_{n+1}-a_n)}{n(n+1)}$

≥ 0

Therefore, $b_{n+1} \geq b_n$.

Therefore (b_n) is monotonic increasing.

The proof is similar if (a_n) is monotonic decreasing.

Convergent sequences

Definition. A sequence $(a_n)$is said to converge to a number $l$if given ϵ> 0 there exists a positive

integer m such that $a_n - l < \epsilon$ $for\ all\ n \geq m$. We say that is the limit of the sequence and we write

$\lim_{n\to\infty} a_n = l$ or $(a_n) \to l$

Note.1 $(a_n) \to l$ iff given ϵ> 0 there exists a natural number m such that $a_n \in (l - \epsilon, l + \epsilon,)$ for all n ≥ m i.e, All but a finite number of terms of the sequence lie within the interval $(l - \epsilon, l + \epsilon)$.

Theorem. 2.1
A sequence cannot converge to two different limits.

Proof. Let (a_n) be a convergent sequence.

If possible let l_1 and l_2 be two distinct limits of (a_n).

Let $\epsilon > 0$ be given.
Since (a_n) →l_1 , there exists a natural number n_1
Such that $|a_n - l_1| < \frac{1}{2}\epsilon$ for all $n \geq n_1$..............(1)
Since (a_n) → l_2, there exists a natural number n_2
Such that $|a_n - l_2| < \frac{1}{2}\epsilon$ for all $n \geq n_2$.................. (2)

Let m = max {n_1 , n_2}
Then $|l_1 - l_2| = |l_1 - a_m + a_m - l_2|$
$\leq |a_m - l_1| + |a_m - l_2|$
$< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon$ by (1) and (2)
$= \epsilon$

∴$l_1 - l_2 < \epsilon$ and this is true for every $\epsilon > 0$. Clearly this is possible only if $l_1 - l_2 = 0$.

Hence $l_1 = l_2$

Examples

1. $\lim_{n\to\infty} \frac{1}{n} = 0$

Proof:

Let ε>0 be given.

Then $\left|\frac{1}{n} - o\right| = \frac{1}{n} < \epsilon$ if $n > \frac{1}{\epsilon}$.Hence if we choose m to any natural number such that m $> \frac{1}{\epsilon}$ then $\left|\frac{1}{n} - o\right| < \epsilon$ for all n$\geq m.$

$\lim_{n\to\infty} \frac{1}{n} = 0$

> **Note.** If $\epsilon = 1/100$, then m can be chosen to be any natural number greater than100.In this example the choice of m depends on the given $\epsilon$and [1/ ϵ] + 1 is the smallest value of m that satisfies the requirements of the definition.

2. The constant sequence 1, 1, 1, …… converges to 1.

Proof.

Let $\epsilon > 0$ be given

Let the given sequence be denoted by (a_n). Then a_n = 1 for all n.

∴$|a_n - 1| = |1 - 1| = 0 < \epsilon$ for all $n \in$ N.
∴ $|a_n - 1| < \epsilon$ for all $n \geq m$ where m can be chosen to be any natural number.
$\therefore \lim_{n\to\infty} a_n = 1$

Note. In this example, the choice of m does not depend on the given ϵ

3. $\lim_{n\to\infty} \frac{n+1}{n} = 1$

Proof. Let $\epsilon > 0$ be given.

Now, $\left|\frac{n+1}{n} - 1\right| = \left|1 + \frac{1}{n} - 1\right| = \left|\frac{1}{n}\right|$

∴ If we choose m to be any natural number greater than 1/ε we have $\left|\frac{n+1}{n} - 1\right| < \epsilon$ for all n ≥ m. Therefore, $\lim_{n\to\infty} \frac{n+1}{n} = 1$

4. $\lim_{n\to\infty} \frac{1}{2^n} = 0$

Proof.

Let $\epsilon > 0$ *be given*

Then $\left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n} < \frac{1}{n}$ (since $2^n > n$ for all n ε N)

$\left|\frac{1}{2^n} - 0\right| < \varepsilon$ for all n ≥ m where m is any natural number greater than 1/ε

Therefore, $\lim_{n\to\infty} \frac{1}{2^n} = 0$

5. The sequence $((-1)^n)$ is not convergent

Proof.

Suppose the sequence $((-1)^n)$ converges to l

Then, given $\epsilon > 0$, there exists a natural number m such that

$|(-1)^n - l| < \epsilon$ *for all* $n > m.$

∴ $|(-1)^m - (-1)^{m+1}| = |(-1)^m - l + l - (-1)^{m+1}|$

$\leq |(-1)^m - l| + |(-1)^{m+1} - l|$

$< \varepsilon + \epsilon = 2\epsilon$

But $|(-1)^m - (-1)^{m+1}| = 2.$

$\therefore 2 < 2\epsilon$

i.e., $1 < \epsilon$ which is a contradiction since $\epsilon > 0$ is arbitrary.

∴ The sequence $((-1)^n)$ is not convergent.

Theorem:2.2

Any convergent sequence is a bounded sequence.

Proof.

Let (a_n) be a convergent sequence.

Let $\lim_{n\to\infty} a_n = l$

Let $\epsilon > 0$ be given. Then there exists m ϵN such that $|a_n - l| < \epsilon$ *for all* $n \geq m$

∴ $|an| < |l| + \epsilon$ *for all* $n \geq m.$

Now, let $k = max\{|a_1|, |a_2|, \ldots\ldots\ldots, |a_{m-1}|, |l| + \epsilon\}$

Then $|(a_n)| \leq k \; for \; all \; n.$

$\therefore (a_n)$ is a bounded sequence.

Note. The converse of the above theorem is not true. For example, the sequence $((-1)^n)$ is a bounded sequence. However it is not a convergent sequence.

Divergent sequence
Definition: A sequence (a_n) is said to diverge to ∞ if given any real number $k > 0$, there exists m ϵN such that $a_n > k$ for all n ≥ m. In symbols we write (an) →∞or $\lim_{n\to\infty} a_n = \infty$

Note. $(a_n) \to \infty$ if given any real number k > 0 there exists m ϵN such that $a_n \epsilon$(k, ∞) for all n ≥ m

Examples

1. $(n) \to \infty$

Proof: Let k > 0 be any given real number.

Choose m to be any natural number such that m > k

Then n > k for all n ≥ m.

$\therefore (n) \to \infty$

2. $(n^2) \to \infty$

Proof: Let k > 0 be any given real number.

Choose m to be any natural number such that m > √k

Then $n^2 > k$ for all n > m

$\therefore (n^2) \to \infty$

Definition. A sequence (a_n) is said to diverge to −∞ if given any real number k < 0 there exists
m ϵN such that that $a_n < k$ for all n ≥ m. In symbols we write
Lim a_n = − ∞, or $(a_n) \to -\infty$

n→∞

Note. $(a_n) \to -\infty$ iff given any real number k < 0, there exists m ϵN such that
$a_n \, \varepsilon \, (-\infty, k) \; for \; all \; n \geq m$
A sequence (a_n) is said to be **divergent** if either $(a_n) \to \infty$ or $(a_n) \to -\infty$

Theorem. 2.3

$(a_n) \to -\infty$ iff $(-a_n) \to -\infty$

Proof.
Let $(a_n) \to \infty$
Let k < 0 be any given real number. Since $(a_n) \to \infty$ there exists m ϵ N such that $a_n > -k$ for all n ≥ m

$\therefore -a_n < k \; for \; all \; n \geq m$
$\therefore (-a_n) \to -\infty.$

Similarly we can prove that if $(-a_n) \to -\infty$ then $(a_n) \to \infty$.

Theorem. 2.4

If $(a_n) \to \infty$ and $a_n \neq 0$ for all $n \in N$ then $(\frac{1}{a_n}) \to 0$.

Proof. Let $\varepsilon > 0$ be given.

Since $(a_n) \to \infty$, there exists $m \in N$ such that $a_n > 1/\varepsilon$ for all $n \geq m$

$\therefore \frac{1}{a_n} < \epsilon$ for all $n \geq m$

$\left|\frac{1}{a_n}\right| < \epsilon$ for all $n \geq m$

Hence $(\frac{1}{a_n}) \to 0$

Note. The converse of the above theorem is not true. For example, consider the sequence (a_n) where

$A_n = (-1)^n/n$. Clearly $(a_n) \to 0$

Now $(1/a_n) = (n/(-1)^n) = -1, 2, -3, 4, \ldots\ldots\ldots$ which neither converges nor diverges to ∞ or $-\infty$

Thus if a sequence $(a_n) \to 0$, then the sequence $(1/a_n) \to \mathbf{0}$ need not converge or diverge.

Theorem:2.5

If $(a_n) \to 0$ and $(a_n) > 0$ for all $n \in N$, *then* $(\frac{1}{a_n}) \to \infty$

Proof.

Let $k > 0$ be any given real number.

Since $(a_n) \to 0$ there exists $m \in N$ such that $|a_n| < 1/k$ *for all* $n \geq m$

$\therefore a_n < 1/k$ for all $n \geq m$ (since $a_n > 0$)

Therefore $1/a_n > k$ for all $n \geq m$

Hence $(1/a_n) \to \infty$

Theorem:2.6

Any sequence (a_n) diverging to ∞ is bounded below but not bounded above.

Proof.

Let $(a_n) \to \infty$. Then for any given real number $k > 0$ there exists $m \in N$ such that $a_n > k$ for all $n \geq m$… ………… (1)

$\therefore$ k is not an upper bound of the sequence (a_n)

$\therefore (a_n)$ is not bounded above

Now let $l = \min\{a_1, a_2, \ldots a_m, k\}$.

From (1) we see that $a_n \geq l$ for all n.

$\therefore (a_n)$ is bounded below

Theorem:2.7

Any sequence (a_n) diverging to $-\infty$ is bounded above but not bounded below.

Proof is similar to that of the previous theorem

Note 1. The converse of the above theorem is not true. For example, the function

f : ℕ→ℝ defined by

$$f(n)=\begin{cases} 0 \text{ if } n \text{ is odd} \\ \frac{1}{2}n \text{ if } n \text{ is even} \end{cases}$$

Determines the sequence 0,1,0,2,0,3,.............which is bounded below and not bounded above. Also for any real number k > 0, we cannot find a natural number m such that $a_n > k$ for all n $\geq$m.

Hence this sequence does not diverge to∞.

Similarly f: ℕ→ℝ given by $f(n)=\begin{cases} 0 \text{ if } n \text{ is odd} \\ \frac{1}{2}n \text{ if } n \text{ is even} \end{cases}$

Determines the sequence 0, −1, 0, −2, 0, ... which is bounded above and not bounded below. However this sequence does not diverge to − ∞.

Oscillating sequence

Definition: A sequence (a_n) which is neither convergent nor divergent to ∞ or −∞ is said to be an

oscillating sequence. An oscillating sequence which is bounded is said to be finitely oscillating. An oscillating sequence which is unbounded is said infinitely oscillating.

Examples.

1. Consider the sequence $((-1)^n)$. Since this sequence is bounded it cannot to ∞or − ∞ (by theorems). Also this sequence is not convergent . Hence ((−1)) is a finitely oscillating sequence.

2. The function f : ℕ → ℝ defined by

$$f(n) = \begin{cases} 0 \text{ if } n \text{ is odd} \\ \frac{1}{2}(1-n) \text{ if } n \text{ is even} \end{cases}$$ determines the sequence 0 , 1 , −1 , 2 , −2 , 3 , ... The range of this sequence is **Z**. Hence it cannot converge or diverge to ±∞. This sequence is infinitely oscillating.

The Algebra of limits

In this section we prove a few simple theorems for sequences which are very useful in calculating limits of sequences.

Theorem: 2.8

If (a_n) → a and (b_n) → b then $(a_n + b_n)$ → a + b.

Proof:

Let $\epsilon > 0$ be given.

Now $|a_n + b_n - a - b| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b|$.....(1)

Since (a_n) → a , there exist a natural number n_1 such that $|a_n-a|<1/2\ \varepsilon$ for all $n \geq n_1$.........(2)

Since (b_n) → b , there exist a natural number n_2 such that $|b_n-b|<1/2\ \varepsilon$ for all $n \geq n_2$.........(3)

Let m = max$\{n_1, n_2\}$

Then $|a_n + b_n - a - b| < 1/2\epsilon + 1/2\epsilon = \epsilon$ for all $n \geq m$. (by (1),(2)and (3))

$\therefore (a_n + b_n) \rightarrow a + b$.

Note. Similarly we can prove that $(a_n - b_n) \rightarrow a - b$.

Theorem:2.9

If $(a_n) \rightarrow a$ and $k \in \mathbf{R}$ then $(k\, a_n) \rightarrow k\, a$.

Proof:

If k = 0, (ka_n) is the constant sequence 0, 0, 0, …. And hence the result is trivial.

Now , let $k \neq 0$.

Then $|k\ a_n - k\, a| = |k|\ |a_n - a|$ …............. (1)

Let $\epsilon > 0$ be given.

Since (a) → a , there exist $m \in \mathbf{N}$ such that

$|a_n - a| < \varepsilon/|k|$ for all $n \geq m$.................. (2)

$\therefore |ka_n - ka| < \epsilon$ for all $n \geq m$ (by 1 and 2).

$\therefore (ka_n) \rightarrow ka$.

Theorem: 2.10

If $(a_n) \rightarrow a$ and $(b_n) \rightarrow b$ then $(a_n b_n) \rightarrow ab$.

Proof.

Let $\epsilon > 0$ be given.

Now, $|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab|$

$\leq |a_n b_n - a_n b| + |a_n b - ab|$

$= |a_n||b_n - b| + |b|\ |a_n - a|$ ...... (1)

Also, since $(a_n) \rightarrow a$, (a_n) is a bounded sequences.

$\therefore$ There exist a real number $k > 0$ such that $|a_n| \leq k$ for all n. .... (2)

Using (1) and (2) we get

$|a_n b_n - ab| \leq k\ |b_n - b| + |b|\ \ |a_n - a|$ …......... (3)

Now since $(a_n) \rightarrow a$, there exist a natural number n_1 such that

$|a_n - a| > \varepsilon/2|b|$ for all $n \geq n_1$ .... (4)

Since $(b_n) \rightarrow b$, there exist a natural number n_2 such that

$|a_n - a| > \varepsilon/2|b|$ for all $n \geq n_2$ .... (5)

Let $m = \max\{n_1, n_2\}$.

Then $|a_n b_n - ab| < k\ (\varepsilon/2k) + |b|\ (\varepsilon/2|b|) = \varepsilon$ for all $n \geq m$ (by (3),(4)and(5))

Hence $(a_n b_n) \rightarrow ab$

Theorem: 2.11

If $(a_n) \rightarrow a$ and $a_n \neq 0$ for all n and $a \neq 0$ then $(\frac{1}{a_n}) \rightarrow \frac{1}{a}$

Proof:

Let $\epsilon > 0$ be given.

We have $|1/a_n - 1/a| = \left|\frac{a_n - a}{a_n a}\right| = \frac{1}{|a_n||a|}|a_n - a|$ … … … … … (1)

Now, $a \neq 0$. Hence $|a| > 0$

Since $(a_n) \rightarrow a$, there exists $n_1 \varepsilon N$ such that $|a_n - a| < \frac{1}{2}|a| \, for \, all \, n \geq n_1$

Hence $|a_n| > \frac{1}{2}|a| \, for \, all \, n \geq n_1$ ....(2)

Using (1) and (2) we get

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| < \frac{2}{|a|^2}|a_n - a| \; for \, all \, n \geq n_1 \dots \dots \dots (3)$$

Now since $(a_n) \rightarrow a$, there exists $n_2 \varepsilon N$ such that

$|a_n - a| < \frac{1}{2}|a|^2\varepsilon \; for \, all \, n \geq n_2$ ....(4)

Let m = max $\{n_1, n_2\}$.

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| < \frac{2}{|a|^2}\frac{1}{2}|a|^2\varepsilon = \varepsilon for \, all \, n \geq m$$

Therefore $(1/a_n) \rightarrow 1/a$

Corollary:

Let $(a_n) \rightarrow$ a and $(b_n) \rightarrow$ b where $b_n \neq 0$ for all n and b $\neq$0.

Then $\left(\frac{a_n}{b_n}\right) \rightarrow \left(\frac{a}{b}\right)$

Proof:

$\left(\frac{1}{b_n}\right) \rightarrow \left(\frac{1}{b}\right)$ (since If $(a_n) \rightarrow a \; and \; a_n \neq 0 \; forall \; n \; and \; a \neq 0 \; then \; (\frac{1}{a_n}) \rightarrow \frac{1}{a}$)

$\left(\frac{a_n}{b_n}\right) \rightarrow \left(\frac{a}{b}\right)$ (since If $(a_n) \rightarrow$ a and $(b_n) \rightarrow$ b then $(a_n b_n) \rightarrow$ ab)

Theorem: 2.12
If $(a_n) \to$ a then $(|a_n|) \to |a|$.

Proof:
Let $\epsilon > 0$ be given
Now $\big|\,|a_n| - |a|\,\big| \le |a_n - a|$ .... (1)
Since $(a_n) \to a$ there exist $m \in \boldsymbol{N}$ such that $|a_n - a| < \epsilon$ for all $n \ge m$.
Hence from (1) we get $\big|\,|a_n| - |a|\,\big| < \epsilon$ for all $n \ge m$.
Hence $(|a_n|) \to (a)$.

Theorem: 2.13
If $(a_n) \to$ a and $a_n \ge 0$ for all n then $a \ge 0$.
Proof.
Suppose $a < 0$. Then $-a > 0$.
Choose ϵ such that $0 < \epsilon < -a$ so that $a + \epsilon < 0$.
Now , since $(a_n) \to a$, there exist $m \in \mathbf{N}$ such that $|a_n - a| < \epsilon$ for all $n \le m$.
$\therefore a - \epsilon < a_n < a + \epsilon$ for all $n \le m$.
Now, since $a + \epsilon < 0$, we have $a_n < 0$ for all $n \ge m$ which is a contradiction since $a_n \ge 0$.
$\therefore a \ge 0$.
Theorem: 2.14
If $(a_n) \to a$, $(b_n) \to b$ and $a_n \le b_n$ for all n , then $a \le b$.
Proof.
Since $a_n \le b_n$, we have $b_n - a_n \ge 0$ for all n.

Also $(b_n - a_n) \to b - a$ (since If $(a_n) \to$ a and $(b_n) \to$ b then $(a_n + b_n) \to a + b$)
$\therefore b - a \ge 0$
$\therefore b \ge a$.

Theorem: 2.15
If $(a_n) \to l, (b_n) \to l$ and $a_n \le c_n \le b_n$ for all n, then $(c_n) \to l$.
Proof.
Let $\epsilon > 0$ be given.
Since $(a_n) \to l$,there exist $n_1 \in \mathbf{N}$ such that $l - \epsilon < a_n < l + \epsilon$ for all $n \ge n_1$.
Similarly, there exist $n_2 \in \mathbf{N}$ such that $l - \epsilon < b_n < l + \epsilon$ for all $n \ge n_2$.
Let $m = \max\{n_1, n_2\}$.
$\therefore - \epsilon < a_n \le c_n \le b_n < l + \epsilon$ for all $n \ge m$.
$\therefore - \epsilon < c_n < l + \epsilon$ for all $n \ge m$.
$\therefore |c_n - l| < \epsilon$ for all $n \ge m$.
$\therefore (c_n) \to l$.

Theorem:2.16
If $(a_n) \to$ a and $a_n \ge 0$ for all n and $a \ne 0$, then $(\sqrt{a_n}) \to \sqrt{a}$.
Proof.
Since $a_n \ge 0$ for all n, $a \ge 0$ (since If $(a_n) \to$ a and $a_n \ge 0$ for all n then $a \ge 0$)
Now, $|\sqrt{a_n} \to \sqrt{a}| = \left|\frac{a_n - a}{\sqrt{a_n} + \sqrt{a}}\right|$
Since $(a_n) \to a \ne 0$, we obtain $a_n > \frac{1}{2}a$ for all $n \ge n_1$

$\sqrt{a_n} > \sqrt{(\frac{1}{2}a)}$ for all n n_1

$|\sqrt{a_n} - \sqrt{a}| < \frac{\sqrt{2}}{(\sqrt{2}+1)\sqrt{a}} |a_n - a| for\ all\ , n \geq n_1$............(1)

Now, let $\epsilon > 0$ be given.

Since (a_n) → a, there exist $n_2 \in$ **N** such that

$|a_n - a| < \epsilon \sqrt{a}\ (\sqrt{2} + 1)/\sqrt{2}$ for all n ≥ n_2 ....(2)

Let m = max {n_1, n_2}.

Then $|\sqrt{a_n} - \sqrt{a}| < \varepsilon$ for all n ≥ m (by 1 and 2).

$\therefore (\sqrt{a_n}) \to \sqrt{a}$.

Theorem: 2.17

If (a_n) → ∞ and (b_n) → ∞ then $(a_n + b_n) \to \infty$.

Proof.

Let k > 0 be any given real number.

Since (a_n) → ∞, there exists $n_1 \in$ **N** such that $a_n > \frac{1}{2}k$ for all n ≥ n_1.

Similarly there exists $n_2 \in$ **N** such that $b_n > \frac{1}{2}k$ for all n ≥ n_2.

Let m = max {n_1, n_2}.

Then $a_n + b_n > k$ for all n ≥ m.

∴ $(a_n + b_n)$ → ∞.

Theorem: 2.18

If (a_n) → ∞ and (b_n) → ∞ then $(a_n b_n)$ → ∞.

Proof.

Let k > 0 be any given real number.

Since (a_n) → ∞, there exist $n_1 \in$ **N** such that $a_n > \sqrt{k}$ for all n ≥ n_1.

Similarly there exists $n_2 \in$ **N** such that $b_n > \sqrt{k}$ for all n ≥ n_2.

Let m = max{n_1, n_2}.

Then $a_n b_n > k$ for all n ≥ m.

∴ $(a_n b_n)$ → ∞.

Theorem: 2.19

Let (a_n) → ∞ then

(i)If c >0 , (c a_n) → ∞

(ii)If c < 0 , (c a_n) → −∞

Proof.

(i) Let c > 0.

Let k > 0 be any given real number.

Since (a_n) → ∞, there exist m ∈N such that $a_n > k/c$ for all n ≥ m.

∴ c $a_n > k$ for all n ≥ m.

∴ (c a_n) → ∞.

(ii) Let c < 0 .

Let k < 0 be any given real number. Then k/c > 0.

∴ There exists m ∈**N** such that $a_n > k/c$ for all n ≥ m.

∴ c $a_n < k$ for all n ≥ m (since c < 0).

∴ (ca_n) → − ∞.

Theorem: 2.20
If $(a_n) \to \infty$ and (b_n) is bounded then $(a_n + b_n) \to \infty$.
Proof.
Since (b_n) is bounded, there exists a real number m < 0 such that $b_n > m$ for all n.
.... (1)
Now, let k > 0 be any real number.
Since m < 0 , k – m > 0.
Since $(a_n) \to \infty$, there exists $n_0 \in \mathbf{N}$ such that $a_n > k - m$ for all $n \geq n_0$ .... (2)
$\therefore a_n + b_n > k - m + m = k$ for all $n \geq n_0$ (by 1 and 2).
$\therefore (a_n + b_n) \to \infty$.

Solved Problems.

1. Show that $\lim_{n\to\infty} \frac{3n^2+2n+5}{6n^2+4n+7} = \frac{1}{2}$

Solution:

$$\frac{3n^2+2n+5}{6n^2+4n+7} = \frac{3+\frac{2}{n}+\frac{5}{n^2}}{6+\frac{4}{n}+\frac{7}{n^2}}$$

Now, $\lim_{n\to\infty}(3+\frac{2}{n}+\frac{5}{n^2}) = 3 + 2\lim_{n\to\infty}\frac{1}{n} + 5\lim_{n\to\infty}\frac{1}{n^2} = 3+0+0 = 3$

Similarly, $\lim_{n\to\infty}(6+\frac{4}{n}+\frac{7}{n^2}) = 6$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{3+\frac{2}{n}+\frac{5}{n^2}}{6+\frac{4}{n}+\frac{7}{n^2}}$$

= 3/6

= ½

2. Show that $\lim_{n\to\infty}(\frac{1^2+2^2+\cdots.......n^2}{n^3}) = \frac{1}{3}$

Solution:

We know that $1^2+2^2+\cdots....+n^2 = \frac{n(n+1)(2n+1)}{6}$

$$\lim_{n\to\infty}(\frac{1^2+2^2+\cdots......n^2}{n^3}) = \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}$$

$= \lim_{n\to\infty}\frac{1}{6}\left(1+\frac{1}{n}\right)(2+\frac{1}{n})$

= 1 / 3

3. Show that $\lim_{n\to\infty}\frac{n}{\sqrt{(n^2+1)}} = 1$

Solution:

$$\lim_{n\to\infty}\frac{n}{\sqrt{(n^2+1)}} = \lim_{n\to\infty}\frac{1}{\sqrt{(1+\frac{1}{n^2})}}$$

$$\frac{1}{\lim_{n\to\infty}\frac{1}{\sqrt{(1+\frac{1}{n^2})}}}$$

$$\frac{1}{\sqrt{\lim_{n\to\infty}(1+\frac{1}{n^2})}}$$

=

=

= 1

4. Show that if $(a_n) \to 0$ and (b_n) is bounded, then $(a_n b_n) \to 0$.

Solution.

Since (b_n) is bounded, there exists k > 0 such that $|b_n| \leq k$ for all n.

$\therefore |a_n b_n| \leq k\,|a_n|$.

Now, let $\epsilon > 0$ be given.

Since $(a_n) \to 0$ there exists m $\in$ **N** such that $|a_n| < \varepsilon/k$ for all $n \geq m$

$\therefore |a_n b_n| < \epsilon$ for all n ≥ m.

$\therefore (a_n b_n) \to 0$.

5. Show that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$

Solution:

$|\sin n| \leq 1$ for all n.

$\therefore (\sin n)$ is a bounded sequences

Also, $(1/n) \to 0$

$\therefore \left(\frac{\sin n}{n}\right) \to 0$ (by problem 4).

6. Show that $\lim(a^{1/n}) = 1$ where a > 0 is any real number.

$n \to \infty$

Solution.

Case (i) Let a = 1 . Then $a^{1/n} = 1$ for each n . Hence $(a^{1/n}) \to 1$

Case (ii) Let a > 1. Then $a^{1/n} > 1$.

Let $a^{1/n} = 1 + h_n$ where $h_n > 0$.

Therefore $a = (1 + h_n)^2$

$= 1 + nh_n + \ldots\ldots . + h_n^n$

$> 1 + nh_n$

Therefore, $h_n < a - 1/n$

Therefore, $0 < h_n < a - 1/n$

Hence $\lim_{n\to\infty} h_n = 0$

Therefore, $(a^{1/n}) = (1 + h_n) \to 1.$

Case(iii)

Let 0<a <1

Then 1/a >1

Therefore, $(1/a)^{1/n} \to 1\ (By\ case\ (i))$

$(\frac{1}{a^{\frac{1}{n}}}) \to 1$

$(a^{1/n})^n \to 1$

7. Show that $\lim_{n\to\infty}(n)^{1/n} = 1$.

Solution.

Clearly $n^{1/n} \geq 1$ for all n.

Let $n^{1/n} = 1 + h_n$ where $h_n \geq 0$

Then $n = (1+h_n)^n$

$= 1 + nh_n + nc_2 h_n^2 + \ldots\ldots\ldots + h_n^n$

$= \frac{1}{2}n(n-1)h_n^2$

Therefore, $h_n^2 < \frac{2}{(n-1)}$

$h_n < \sqrt{\frac{2}{n-1}}$

Since $\sqrt{\frac{2}{n-1}} \to 0$ and $h_n \geq 0, (h_n) \to 0$

Hence $(n^{1/n}) = (1+h_n) \to 1$.

8. Give an example to show that if (a_n) is a sequence diverging to ∞ and (b_n) is sequence diverging to $-\infty$ then $(a_n + b_n)$ need not be a divergent sequence.

Solution.

Let $(a_n) = (n)$ and $(b_n) = (-n)$.

Clearly $(a_n) \to 0$ and $(b_n) \to -\infty$.

However $(a_n + b_n)$ is the constant sequence 0 , 0 , 0 ,.... Which converges to 0.

UNIT - III

BEHAVIOUR OF MONOTONIC SEQUENCES

Theorem: 3.1

i. A monotonic increasing sequence which is bounded above converges to its l.u.b.
ii. A monotonic increasing sequence which is bounded above diverges to ∞.
iii. A monotonic decreasing sequence which is bounded below converges to its g.l.b.
iv. A monotonic decreasing sequence which is bounded below diverges to -∞.

Proof:

(i) Let (a_n) be a monotonic increasing sequence which is bounded above.

Let k be the l.u.b of the sequence.

Then $a_n \leq k$ for all n.

Let ε>0 be given

Therefore, k-ε <k and hence k-ε is not an upper bound of (a_n)

Hence, there exists a_n such that $a_m > k - \varepsilon$.

Now, since (a_n) is monotonic increasing $a_n \geq a_m$ for all n > m

Hence a_n >k-ε for all n≥m...........(2)

Therefore $k - \varepsilon < a_n \leq k$ for all n ≥m.(by 1 and 2)

Therefore $|a_n - k| < \varepsilon$ for all n ≥ m.

Therefore $(a_n) \to k$.

(ii) Let (a_n) be a monotonic increasing sequence which is not bounded above.

Let k > 0 be any real number.

since (a_n) is not bounded, there exists m ε N such that $a_m > k$.

Also $a_n \geq a_m$ for all n ≥m.

$\therefore a_n > k \; for \; all \; n \geq m$

$Hence,$ $(a_n) \to \infty$

Proof of (iii) is similar to that of (i)

Proof of (iv) is similar to that of (ii)

Note:

The above theorem shows that a monotonic sequence either converges or diverges. Thus a Monotonic sequence cannot be an oscillating sequence.

Solved Problems:

1. Let $a_n = 1+\frac{1}{1!} + \frac{1}{2!} + \cdots \ldots + \frac{1}{n!}$. Show that $\lim_{n\to\infty} a_n$ exists and lies between 2 and 3.

Solution:

Clearly $(a_n)$is a monotonic increasing sequence

Also, $a_n = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots\ldots+\frac{1}{n!}$.

$\leq 1+\ 1+\frac{1}{2}+\frac{1}{2^2}+\cdots\ldots+\frac{1}{2^{n-1}}$.

$= 1+\left(\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}\right)$

$= 1+ 2\left(1-\frac{1}{2^n}\right)$

$= 3 - \frac{1}{2^{n-1}} < 3$

$\therefore\ a_n < 3$

$\therefore\ (a_n)$is bounded above

$Therefore, \lim_{n\to\infty} a_n$ exi ts

$Also$ $2< a_n <3$ for all n.

$\therefore 2 < \lim_{n\to\infty} a_n < 3$

Hence the result.

1. Show that the sequence $(1+\frac{1}{n})^n$ converges.

Solution:

Let $a_n=(1+\frac{1}{n})^n$

By binomial theorem,

$a_n = 1+1+\left(\frac{n(n-1)}{2!}\right)\frac{1}{n^2}+\left(\frac{n(n-1)(n-2)}{3!}\right)\frac{1}{n^3}+\cdots\ldots\ldots+\frac{1}{n^n}$

$1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots\ldots\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots(1-\frac{n-1}{n})$

$< 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\ldots+\frac{1}{n!}$

< 3 (by problem 1)

Therefore,(a_n) is bounded above.

Also,

$a_{n+1}=1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)+\cdots\ldots\ldots+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\ldots\ldots(1-\frac{n}{n+1})$

$>1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots\ldots\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\ldots\ldots(1-\frac{n-1}{n})$

$\therefore\ a_{n+1} > a_n$

$\therefore\ (a_n)$ is monotonic increasing.

$\therefore\ (a_n)$ is a convergent sequence.

Theorem: 3.2 (Cauchy's First Limit Theorem)

If $(a_n) \to l\ then\ \left(\frac{a_1+a_2+\cdots\ldots+a_n}{n}\right) \to l.$

Proof:

Case (i).

Let $l = 0$

Let $b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$

Let ε>0 be given.

Since $(a_n) \to 0$ there exists m ϵ N such that $|a_n| < (1/2)\varepsilon$ for all $n \geq m$.........(1)

Now let $n \geq m$

Then $|b_n| = \left|\frac{a_1 + a_2 + \cdots + a_m + a_{m+1} + \cdots + a_n}{n}\right|$

$\leq \frac{|a_1| + |a_2| + \cdots + |a_m|}{n} + \frac{|a_{m+1}| + \cdots + |a_n|}{n}$

$= \frac{k}{n} + \frac{|a_{m+1}| + \cdots + |a_n|}{n}$ where $k = |a_1| + |a_2| + \cdots + |a_m|$

$< \frac{k}{n} + \left(\frac{n-m}{n}\right)\frac{\varepsilon}{2}$ (by (1))

$< \frac{k}{n} + \frac{\varepsilon}{2}$ (Since $\frac{n-m}{n} < 1$)… ..... (2)

Now since $(k/n) \to 0$, there exists $n_0 \epsilon N$ such that $k/n < (1/2)\varepsilon$ for all $n \geq n_0$............ (3)

Let $n_1 = \max\{m, n_0\}$

Then $|b_n| < \varepsilon$ for all $n \geq n_1$ (using 2 and 3)

Therefore $(b_n) \to 0$

Case (ii)

Let $l \neq 0$

Since $(a_n) \to l$, $(a_n - l) \to 0$

$\therefore \left(\frac{(a_1 - l) + (a_2 - l) + \cdots + (a_n - l)}{n}\right) \to 0$ (by case (i))

$\therefore \left(\frac{a_1 + a_2 + \cdots + a_n - nl}{n}\right) \to 0$

$\therefore \left(\frac{a_1 + a_2 + \cdots + a_n}{n} - l\right) \to 0$

$\therefore \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \to l$

Theorem: 3.3 (Cesaro's theorem)

If $(a_n) \to a$ and $(b_n) \to b$ then $\left(\frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n}\right) \to ab$

Proof:

Let $c_n = \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n}$

Now put $a_n = a + r_n$ so that $(r_n) \to 0$

Then $c_n = \frac{(a + r_1) b_n + \cdots + (a + r_n) b_1}{n}$

$= \frac{a(b_1+\cdots+b_n)}{n} + \frac{r_1 b_n+\cdots+r_n b_1}{n}$

Now, by Cauchy's first limit theorem, $\left(\frac{b_1+\cdots+b_n}{n}\right) \to l$

$$\therefore \left(\frac{a(b_1+b_2+\cdots..+b_n)}{n}\right) \to ab$$

Hence it is enough if we prove that $\left(\frac{r_1 b_n+\cdots+r_n b_1}{n}\right) \to 0$

Since, since $(b_n) \to b$, (b_n) is a bounded sequence.

Therefore, there exists a real number k>0 such that $|b_n| \le k$ *for all n.*

$$\therefore \left|\frac{r_1 b_n+\cdots..+r_n b_1}{n}\right| \le k\left|\frac{r_1+\cdots..+r_n}{n}\right|$$

Since $(r_n) \to 0$, $\left(\frac{r_1 b_n+\cdots+r_n b_1}{n}\right) \to 0$

$\left(\frac{r_1 b_n+\cdots+r_n b_1}{n}\right) \to 0$

Hence the theorem.

Theorem: 3.4 (Cauchy's Second Limit Theorem)

Let (a_n) be a sequence of positive terms. Then $\lim_{n\to\infty} a_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ provided the limit on the right hand side exists, whether finite or infinite.

Proof:

Case(i) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}=1$, finite.

Let ε>0 be any given real number.

Then there exists m∊ N such that $l-\frac{1}{2}\varepsilon < \frac{a_{n+1}}{a_n} < l+\frac{1}{2}\varepsilon$ *for all* $n \ge m$

Now choose $n \ge m$

Then $l-\frac{1}{2}\varepsilon < \frac{a_{m+1}}{a_m} < l+\frac{1}{2}\varepsilon$

$$l-\frac{1}{2}\varepsilon < \frac{a_{m+2}}{a_{m+1}} < l+\frac{1}{2}\varepsilon$$

....

....

$$l-\frac{1}{2}\varepsilon < \frac{a_n}{a_{n-1}} < l+\frac{1}{2}\varepsilon$$

Multiplying these inequalities, we obtain

$$\left(l-\frac{1}{2}\varepsilon\right)^{n-m} < \frac{a_n}{a_m} < \left(l+\frac{1}{2}\varepsilon\right)^{n-m}$$

$$\therefore a_m \frac{\left(l-\frac{1}{2}\varepsilon\right)^n}{\left(l-\frac{1}{2}\varepsilon\right)^m} < a_n < a_m < a_m \frac{\left(l+\frac{1}{2}\varepsilon\right)^n}{\left(l+\frac{1}{2}\varepsilon\right)^m}$$

$\therefore k_1\left(l-\frac{1}{2}\varepsilon\right)^n < a_n < k_2\left(l+\frac{1}{2}\varepsilon\right)^n$, Where k_1, k_2 are some constants

$\therefore k_1^{\frac{1}{n}}(l-\frac{1}{2}\varepsilon) < a_n^{\frac{1}{n}} < k_2^{\frac{1}{n}}\left(l+\frac{1}{2}\varepsilon\right)$ ....(1)

Now, $(k_1^{\frac{1}{n}}(l-\frac{1}{2}\varepsilon)) \to l-\frac{1}{2}\varepsilon$ (Since $(k_1^{\frac{1}{n}}) \to l$)

$\therefore$ There exists $n_1 \in N$ such that $(l-\frac{1}{2}\varepsilon)-\frac{1}{2}\varepsilon < k_1^{\frac{1}{n}}(l-\frac{1}{2}\varepsilon) < (l-\frac{1}{2}\varepsilon)+\frac{1}{2}\varepsilon$

$for\ all\ n \geq n_1$ ......(2)

Similarly, there exists $n_2 \in N$ such that $(l+\frac{1}{2}\varepsilon)-\frac{1}{2}\varepsilon < k_2^{\frac{1}{n}}(l+\frac{1}{2}\varepsilon) < (l+\frac{1}{2}\varepsilon)+\frac{1}{2}\varepsilon$

$for\ all\ n \geq n_2$ ......(3)

Let n_0 = max {m, n_1,n_2}.

Then $l-\varepsilon < k_1^{\frac{1}{n}}(l-\frac{1}{2}\varepsilon) < a_n^{\frac{1}{n}} < k_2^{\frac{1}{n}}(l+\frac{1}{2}\varepsilon) < l+\varepsilon for\ all\ n \geq n_0$ $(by\ 1,2\ and\ 3)$

$\therefore l-\varepsilon < a_n^{\frac{1}{n}} < l+\varepsilon\ for\ all\ n \geq n_0$

Hence $(a_n^{\frac{1}{n}}) \to l.$

Case (ii):

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\infty$$

Then $\lim_{n\to\infty}(\frac{1}{a_{n+1}})/(\frac{1}{a_n})=0$

Therefore, By case (i), $(\frac{1}{a_n})^{\frac{1}{n}} \to 0$

Hence $(a_n^{\frac{1}{n}}) \to \infty$

Theorem: 3. 5

Let (a_n)be any sequence and $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = l$. If $l>1, then\ (a_n) \to 0.$

Theorem: 3.6

Let (a_n)be any sequence of positive terms and $\lim_{n\to\infty}\left(\frac{a_n}{a_{n+1}}\right) = l$. If $l<1, then\ (a_n) \to \infty.$

Problems:

1. Show that $\lim_{n\to\infty}\frac{1}{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)=0$

Solution:

Let a_n= 1/n

We know that $(a_n) \to 0$. Hence by Cauchy's first limit theorem we get

$$\left(\frac{a_1+a_2+\cdots..+a_n}{n}\right) \to 0$$

2. Show that $\lim_{n\to\infty}\frac{n!}{n^n}=0$

Solution:

Let $a_n = \frac{n!}{n^n}$

$$\therefore \left|\frac{a_n}{a_{n+1}}\right| = \frac{n!}{n^n}\frac{(n+1)^{n+1}}{(n+1)!}$$

$= (\frac{n+1}{n})^n$

$= (1 + \frac{1}{n})^n$

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} (1 + \frac{1}{n})^n$$

$= e > 1$

Hence $(a_n) \to 0$

Subsequence

Definition. Let (a_n) be a sequence. Let (a_{n_k}) be a strictly increasing sequence of natural numbers. Then (a_{n_k}) is called a subsequence of (a_n).

Note. The terms of a subsequences occur in the same order in which they occur in the original sequence.

__**Examples.**__

1. (a_{2n}) is a subsequence of any sequence (a_n). Note that in this example the interval between any two terms of the subsequence is the same, (i.e.,) $n_1=2$, $n_2=4$, $n_3=6$,... $n_k= 2k$.
2. (a_{n^2}) is a subsequence of any sequence (a_n). Hence $a_{n_1} = a_1$, $a_{n_2} = a_4$, $a_{n_3} = a_9$ ..... Here the interval
between two successive terms of the subsequence goes on increasing as k becomes large. Thus the interval between various terms of a subsequence need not be regular.
3. Any sequence (a_n) is a subsequence of itself.

Theorem: 3.7

If a sequence (a_n) converges to l , then every subsequence (a_{n_k}) of (a_n) also converges to l.

Proof.

Let $\epsilon > 0$ be given.
Since $(a_n) \to l$ there exists $m \in \boldsymbol{N}$ such that
$|a_n - l| < \epsilon$ for all $n \geq m$. ....(1)
Now choose $n_{k_0} \geq m$.
Then $k \geq k_0 \Rightarrow n_k \geq n_{k_0}$ ($\because$ (n_k) is monotonic increasing)
$\Rightarrow n_k \geq m$.
$\Rightarrow |a_{n_k} - l| < \epsilon$ (by 1)
Thus $|a_{n_k} - l| < \epsilon$ for all $k \geq k_0$.
$\therefore (a_{n_k}) \to l$.
Note 1. If a subsequence of a sequence converges, then the original sequence need not converge.

Theorem :3.8

If the subsequences (a_{2n-1}) and (a_{2n}) of a sequence (a_n) converge to the same limit l

then (a_n) also converges to l .

Proof.

Let $\epsilon > 0$ be given. Since $(a_{2n-1}) \to l$ there exists $n_1 \in \boldsymbol{N}$ such that $|a_{2n-1} - l| < \epsilon$ for all $2n - 1 \geq n_1$.

Similarly there exists $n_2 \in \boldsymbol{N}$ such that $|a_{2n} - l| < \epsilon$ for all $2n \geq n_2$.

Let m = max$\{n_1, n_2\}$.
Clearly $|a_n - l| < \epsilon$ for all n ≥ m.
$\therefore (a_n) \to l$.

Note. The above result is true even if we have $l \to \infty$ or $-\infty$.

Definition. Let (a_n) be a sequence. A natural number m is called a **peak point** of the sequence (a_n)
if $a_n < a_m$ for all n > m.

Example.

1. For the sequence (1/n), every natural number is a peak point and hence the sequence has infinite number of peak point. In general for a strictly monotonic decreasing sequence every natural number is a peak point.
2. Consider the sequence 1 , ½,1/3, -1, -1,….. Here 1 , 2, 3 are the peak points of the sequence.
3. The sequence 1 , 2 , 3 , …… has no peak point. In general a monotonic increasing sequence has no Peak point.

Theorem :3.9

Every sequence (a_n) has no monotonic subsequence.
Proof.
Case (i)
(a_n) has infinite number of peak points. Let the peak points be
$n_1 < n_2 <$ …. $< n_k <$ ……
Then $a_{n1} > a_{n2} >$ ….. $> a_{nk} >$ ….
$\therefore (a_{n_k})$ is a monotonic decreasing subsequence of (a_n).

Case (ii)

(a_n) has only a finite number of peak points or no peak points.
Choose a natural number n_1 such that there is no peak point greater than or equal to n_1.
Since n_1 is not a peak point of (a_n) , there exists $n_2 > n_1$ such that $a_{n2} \geq a_{n1}$.
Again since n_2 is not a peak point , there exist $n_3 > n_2$ such that $a_{n3} \geq a_{n2}$.
Repeating this process we get a monotonic increasing subsequence (a_{n_k}) of (a_n).

Theorem : 3.10

Every bounded sequences has a convergent subsequences.

Proof.
Let (a_n) be a bounded sequence. Let (a_{n_k}) be monotonic subsequence of (a_n) .
since (a_n) is bounded, (a_{n_k}) is also bounded.

∴ (a_{n_k}) is a bounded monotonic sequence and hence converges.

∴ (a_{n_k}) is a convergent subsequence of (a_n).

<u>Cauchy sequences.</u>
Definition. A sequence (a_n) is said to be a **Cauchy sequence** if given ϵ> 0, there exists $n_0 \in \boldsymbol{N}$ such that
$|a_n - a_m| < \epsilon$ for all n , m ≥ n_0.

Note. In the above definition the condition $|a_n - a_m| < \epsilon$ for all n , m ≥ n_0 can be written in the

following equivalent form, namely , $|a_{n+p} - a_n| < \epsilon$ for all n ≥ n_0 and for all positive integers p.

<u>Examples</u>

1. The sequence (1/n) is a Cauchy sequence.

Proof.

Let(a_n) =(1/n).
Let ϵ>0 be given.
Now, $|a_n - a_m|$=|1/n -1/m|
∴ If we choose n_0 to be any positive integer greater than 1/ε , we get
$|a_n - a_m| \leq \epsilon$ for all n, m ≥ n_0.
∴ (1/n) is a Cauchy sequence.

2. The sequence ($(-1)^n$) is not a Cauchy sequence.
Proof.
Let (a_n) = ($(-1)^n$) .
∴ $|a_n - a_{n+1}|$ = 2.
∴If $\boldsymbol{\epsilon}$<2 , we cannot find n_0 such that $|a_n - a_{n+1}| < \epsilon$ for all n ≥n_0.
∴($(-1)^n$) is not a Cauchy sequence.
3. (n) is not a Cauchy sequence.
Proof.
Let (a_n) = (n).
∴ $|a_n - a_m| \geq 1$ if n ≠ m.
∴ If we choose $\boldsymbol{\epsilon}$< 1 , we cannot find n_0 such that $|a_n - a_m| < \epsilon$ for all n , m ≥ n_0.
∴ (n) is not a Cauchy sequence.

Theorem :3.11
Any convergent sequence is a Cauchy sequence.
Proof.

Let $(a_n) \to l$. Then given $\epsilon > 0$, there exists $n_0 \in \boldsymbol{N}$ such that $|a_n - l| < (1/2)\varepsilon$ for all $n \geq n_0$

$\therefore |a_n - a_m| = |a_n - l + l - a_m|$

$\leq |a_n - l| + |l - a_m|$

$< (1/2)\varepsilon + (1/2) = \varepsilon$ for all n , m $\geq n_0$.

$\therefore (a_n)$ is a Cauchy sequence.

Theorem .3.12

Any Cauchy sequence is a bounded sequence .

Proof.

Let (a_n) be a Cauchy sequence.

Let $\epsilon > 0$ be given. Then there exists $n_0 \in \boldsymbol{N}$ such that $|a_n - a_m| < \epsilon$ for all n , m $\geq n_0$.

$\therefore |a_n| < |a_{n0}| + \varepsilon$ for $n \geq n_0$.

Now , let k = max { $|a_1|, |a_2|, \ldots. |a_{n0}| + \varepsilon$ } .

Then $|a_n| \leq k$ for all n.

$\therefore (a_n)$ is a bounded sequence.

Theorem . 3.13

Let (a_n) be a Cauchy sequence. If (a_n) has a subsequence (a_{n_k}) converging to , then $(a_n) \to l$.

<u>Proof.</u>

Let $\epsilon > 0$ be given. Then there exists $n_0 \in \boldsymbol{N}$ such that

$|a_n - a_m| < (1/2)\,\epsilon$ for all n , m $\geq n_0$....(1)

Also since $(a_{n_k}) \to l$, there exists $k_0 \in \boldsymbol{N}$ such that $|a_{n_k} - l| < \frac{1}{2}\varepsilon$ *for all* $k \geq k_0$ ......(2)

Choose n_k such that $n_k > n_{k0}$ and n_0

Then $|a_n - l| = |a_n - a_{nk} + a_{nk} - l|$

$\leq |a_n - a_{nk}| + |a_{nk} - l|$

$= (1/2)\,\epsilon + (1/2)\varepsilon$

$= \varepsilon$ for all $n \geq n_k$.

Hence $(a_n) \to l$.

<u>Theorem : 3.14 (Cauchy's General Principle of Convergence Sequence)</u>

A sequence (a_n) in **R** is convergent iff it is a Cauchy sequence.

Proof.

we have proved that any convergent sequence is a Cauchy sequence.

Conversely, let (a_n)be a Cauchy sequence in **R.**

$\therefore (a_n)$ is a bounded sequence (Any Cauchy sequence is a bounded sequence)

$\therefore$ There exist a subsequence (a_{n_k}) of (a_n) such that $(a_{n_k}) \to l$

$\therefore (a_n) \to l$ (by previous theorem).

UNIT - IV
SERIES

Infinite series

Definition. Let (a_n) = a_1, a_2 , ….. a_n , ….. be a sequence of real numbers. Then the formal expression a_1+a_2 + …..+ a_n +….. is called an infinite series of real numbers and is denoted by $\sum_1^\infty a_n$ or $\sum a_n$

Let $s_1 = a_1$; $s_2 = a_1 + a_2$; $s_3 = a_1 + a_2 + a_3$;…. $s_n = a_1 + a_2 + \cdots + a_n$.

Then (s_n) is called the sequence of partial sums of the given series $\sum a_n$.

The series $\sum a_n$ is said to converge, diverge or oscillate according as the sequence of partial sums (s_n) converges, diverges or oscillates.

If $(s_n) \to s$, we say that the series $\sum a_n$ converges to the sum s.
We note that the behavior of a series does not change if a finite number of terms are added or altered.

Examples.

Consider the series 1 + 1 + 1 + 1…… Here s_n= n. Clearly the sequence (s_n) diverges to ∞.
Hence the given series diverges to ∞.

2. Consider the geometric series $1 + r + r^2 + \ldots\ldots + r^n + \ldots$

Here, $s_n = 1 + r + r^2 + \ldots\ldots + r^{n-1} = \frac{1-r^n}{1-r}$.

Case (i) 0 < r < 1. Then $(r^n) \to 0$

Therefore, $(s_n) \to \frac{1}{1-r}$ ∴ The given series converges to the sum 1/(1-r)

Case (ii) r > 1.

Then $s_n = \frac{r^n - 1}{r-1}$

Also $(r^n) \to \infty$ when r > 1

Hence the series diverges to ∞

Case (iii) r = 1.

Then the series becomes 1 + 1 + ….
$(s_n) = (n)$. which diverges to ∞.
Case (iv) r = −1.
Then the series becomes 1 −1 + 1 −1 + ……

$$\therefore s_n = \begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ if } n \text{ is odd} \end{cases}$$

∴ (s_n) oscillates finitely.
Hence the given series oscillates finitely.
Case (v) : r < −1.

∴ (r^n) oscillates infinitely
∴ (s_n) oscillates infinitely.
Hence the given series oscillates infinitely.

Note 1. Let $\sum a_n$ be a series of positive terms. Then (s_n) is a monotonic increasing sequence. Hence (s_n) converges or diverges to ∞ according as (s_n) is bounded or unbounded. Hence the series $\sum a_n$ converges or diverges to ∞. Thus a series of positive terms cannot oscillate.

Note 2. Let $\sum a_n$ be a convergent series of positive terms converging to the sum s. Then s is the l. u. b. of (s_n). Hence $s_n \leq s$ for all n.
Also given $\epsilon > 0$ there exists m ∈**N** such that $s - \epsilon < s_n$ for all $n \geq m$.
Hence $s - \epsilon < s_n \leq s$ for all $n \leq m$.

Theorem : 4.1
Let $\sum a_n$ be a convergent series converging to the sum s .
Then $\lim_{n\to\infty} a_n = 0$
Proof.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1})$$

$$= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}$$

$$= s - s = 0.$$

Theorem . 4.2

Let $\sum a_n$ converge to a and $\sum b_n$ converge to b . Then $\sum(a_n \pm b_n)$ converges to $a \pm b$ and $\sum ka_n$ converges to ka.

Proof.

Let $s_n = a_1 + a_2 + \ldots + a_n$ and $t_n = b_1 + b_2 + \ldots + b_n$. Then $(s_n) \to a$ and $(t_n) \to b$.
∴ $(s_n \pm t_n) \to a \pm b$
Also $(s_n \pm t_n)$ is the sequence of partial sums of $\sum(a_n \pm b_n)$

∴ $\sum(a_n \pm b_n)$ converges to $a \pm b$.
Similarly ka_n converges to ka.

<u>**Theorem 4.3 (Cauchy's general principle of convergence in Series)**</u>
The series $\sum a_n$ is convergent iff given $\epsilon > 0$ there exists $n_0 \in$ **N** such that $|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \epsilon$ for all $n \geq n_0$ and for all positive integers p.

Proof.

Let $\sum a_n$ be a convergent series. Let $s_n = a_1 + \ldots + a_n$.
∴ (s_n) is a convergent sequence.
∴ (s_n) is a Cauchy sequence
∴ There exists $n_0 \in \mathbf{N}$ such that $|s_{n+p} - s_n| < \epsilon$ for all $n \geq n_0$ and for all $p \in \mathbf{N}$.
∴ $|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \epsilon$ for all $n \geq n_0$ and for all $p \in \mathbf{N}$.
Conversely if $|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \epsilon$ for all $n \geq n_0$ and for all $p \in \mathbf{N}$ then (s_n) is a Cauchy sequence in **R** and hence (s_n) is convergent.
∴ The given series converge.

Solved Problems.

1. Apply Cauchy's general principle of convergence to show that the series $\sum \frac{1}{n}$ not convergent.

Solution. Let $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

Suppose the series $\sum \frac{1}{n}$ is convergent.

∴ By Cauchy's general principle of convergence, given $\epsilon > 0$ there exists $m \in \mathbf{N}$ such that

$|s_{n+p} - s_n| < \epsilon$ for all $n \geq m$ and for all $p \in \mathbf{N}$.

$\left|\left(1 + \frac{1}{2} + \cdots + \frac{1}{n+p}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)\right| < \varepsilon$ for all $n \geq m$ and for all $p \in \mathbf{N}$.

$\left|\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p}\right| < \epsilon$ for all $n \geq m$ and for all $p \in \mathbf{N}$.

In particular if we take $n = m$ and $p = m$ we obtain $\frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{m+m} > \frac{1}{2m} + \cdots + \frac{1}{2m} = \frac{1}{2}$

∴ $\frac{1}{2} < \epsilon$ which is a contradiction since $\epsilon > 0$ is arbitrary.

∴ The given series is not convergent.

Comparison test

Theorem 4.4 (Comparison test)

i). Let Σc_n be a convergent series of positive terms. Let Σa_n be another series of positive terms. If there exists $m \in \mathbf{N}$ such that $a_n \leq c_n$ for all $n \geq m$, then Σa_n is also convergent.

ii). Let Σd_n be a divergent series of positive terms. Let Σa_n be another series of positive terms. If there exists $m \in \mathbf{N}$ such that $a_n \leq d_n$ for all $n \geq m$, then Σa_n is also divergent.

Proof:

(i) Since the convergence or divergence of a series is not altered by the removal of a finite number
of terms we may assume without loss of generality that $a_n \leq c_n$ for all n.

Let $s_n = c_1 + c_2 + \ldots + c_n$ and $t_n = a_1 + a_2 + \ldots + a_n$.

Since $a_n \le c_n$ we have $t_n \le s_n$.
Now, Since Σc_n is convergent, (s_n) is a convergent sequence.
$\therefore (s_n)$ is a bounded sequence.
$\therefore$ There exists a real positive number k such that $s_n \le$ k for all n.
$\therefore t_n \le$ k for all n
Hence (t_n) is bounded above.
Also (t_n) is a monotonic increasing sequence.
$\therefore (t_n)$ converges
$\therefore \Sigma a_n$ converges.

(ii)Let Σd_n diverge and $a_n \ge d_n$ for all n.
$\therefore t_n \ge s_n$.
Now, (s_n) is diverges to ∞.
$\therefore (s_n)$ is not bounded above.
$\therefore (t_n)$ is not bounded above.
Further (t_n) is monotonic increasing and hence (t_n) diverges to ∞.
$\therefore \Sigma a_n$ diverges to ∞.

Theorem :4.5

(i) If Σc_n converges and if $\lim_{n\to\infty}\left(\frac{a_n}{c_n}\right)$ exists and is finite then Σa also converges.

(ii)If $\Sigma \frac{a_n}{c_n} d_n$ diverges and if $\lim_{n\to\infty}\left(\frac{a_n}{d_n}\right)$ exists and is greater than zero then Σa_n diverges.

Proof

(i) .Let $\lim_{n\to\infty}\left(\frac{a_n}{c_n}\right)$=k

Let $\varepsilon > 0$ be given. Then there exists $n \in \mathbf{N}$ such that $\frac{a_n}{c_n} <$ k + ϵ for all n $\ge n_1$.

$\therefore a_n <$ (k + ϵ) c_n for all n $\ge n_1$.
Also since Σc_n is a convergent series, Σ $(k+\epsilon)$ c_n is also convergent series.
$\therefore$ By comparison test Σa_n is convergent.

(ii)Let $\lim_{n\to\infty}\left(\frac{a_n}{c_n}\right)$=k > 0

Choose $\epsilon = \frac{1}{2}k$. Then there exists $n_1 \in \mathbf{N}$ such that k $- \frac{1}{2}k < \frac{a_n}{d_n} <$ k$+ \frac{1}{2}k$ for all n $\ge n_1$.

$\therefore \frac{a_n}{d_n} > \frac{1}{2}k$ for all n $\ge n_1$

$\therefore a_n > \frac{1}{2}k d_n$ for all n $\ge n_1$

Since d_n is a divergent series, $\Sigma \frac{1}{2} k d_n$ is also divergent series.

$\therefore$ By comparison test, Σa_n diverges.

Theorem: 4.6

i) Let Σc_n be a convergent series of positive terms. Let Σa_n be another series of positive terms. If there exists $m \in \mathbf{N}$ such that $\frac{a_{n+1}}{a_n} \le \frac{c_{n+1}}{c_n}$ for all n$\ge$ m, then Σa_n is convergent.

ii) Let Σd_n be a divergent series of positive terms. Let Σa_n be another series of positive terms. If there exists $m \in \mathbf{N}$ such that

$\frac{a_{n+1}}{a_n} \geq \frac{d_{n+1}}{d_n}$ for all n ≥ m, then Σa_n is divergent.

Proof.(i)

$\frac{a_{n+1}}{a_n} \leq \frac{a_n}{c_n}$

$\therefore (\frac{a_n}{c_n})$ is a monotonic decreasing sequence.

$\therefore \frac{a_n}{c_n} \leq$ k for all n where k $= \frac{a_1}{c_1} \therefore a_n \leq kc_n$ for all $n \in \mathbf{N}$.

Now, Σc_n is convergent. Hence Σkc_n is also a convergent series of positive terms.

$\therefore \Sigma a_n$ is also convergent

(ii)Proof is similar to that of (i).

Theorem .:4.7

The harmonic series $\sum \frac{1}{n^p}$ converges if p > 1 and if p ≤ 1.

Proof.

Case (i) Let p=1.

Then the series becomes Σ(1/n) which diverges.

Case (ii) Let p < 1.

Then $n^p < n$ for all n.

$\therefore \frac{1}{n^p} > \frac{1}{n}$ for all n

∴ By comparison test $\sum \frac{1}{n^p}$ diverges.

Case (iii) Let p > 1.

Let $s_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots . + \frac{1}{n^p}$

Then $\overline{S_{2^{n+1}-1}} = 1 + \frac{1}{2^p} + \cdots . + \frac{1}{(2^{n+1}-1)^p}$

$=1+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\cdots.+\left(\frac{1}{(2^n)^p}+\frac{1}{(2^{n+1})^p}+\ \ldots.+\frac{1}{(2^{n+1}-1)^p}\right)$

$<1 + 2(\frac{1}{2^p}) + 4(\frac{1}{4^p}) + \cdots . + 2^n(\frac{1}{(2^n)^p})$

$=1+\frac{1}{2^{p-1}} + \frac{1}{2^{2p-2}} + \frac{1}{2^{(p-1)n}}$

$\therefore \; s_{2^{n+1}-1} < 1 + \frac{1}{2^{p-1}} + (\frac{1}{2^{p-1}})^2 + \cdots .. + (\frac{1}{2^{p-1}})^n$

Now, since p > 1, p-1> 0

Hence

$\frac{1}{2^{p-1}} < 1$

Therefore $1+\frac{1}{2^{p-1}}+(\frac{1}{2^{p-1}})^2+\cdots..+(\frac{1}{2^{p-1}})^n<\frac{1}{1-\frac{1}{2^{p-1}}}$ = k(say)

$\therefore\ s_{2^{n+1}-1}<k$

Now let n be any positive integer. Choose $m\in\mathbf{N}$ such that $n\le 2^{m+1}-1$. Since (s_n) is a monotonic increasing sequence , $s_n\le S_{2^{m+1}-1}$.

Hence $s_n<k$ for all n.

Thus (s_n) is a monotonic increasing sequence and is bounded above.

$\therefore(s_n)$ is convergent.

$\therefore\sum\frac{1}{n^p}$ is convergent.

Solved problems.

1. Discuss the convergence of the series $\sum\frac{1}{\sqrt{(n^3+1)}}$

Solution.

$\frac{1}{\sqrt{(n^3+1)}}<\frac{1}{n^{\frac{3}{2}}}$

Also $\Sigma\ \frac{1}{n^{\frac{3}{2}}}$is convergent

$\therefore$ By comparison test, $\sum\frac{1}{\sqrt{(n^3+1)}}$ is convergent.

2. Discuss the convergence of the series $\sum_3^\infty(\log\log n)^{-\log n}$.

Solution.

Let $a_n=(\log\log n)^{-\log n}$

$\therefore a_n=n^{-\theta_n}$ where $\theta_n=\log(\log\log\ n)$.

Since $\lim_{n\to\infty}\log\log\log n=\infty$, there exists $m\in\mathbf{N}$ such that $\theta_n\ge 2$ for all $n\ge m$.

$\therefore n^{-\theta}\le n^{-2}$ for all $n\ge m$.

$\therefore a_n\le n^{-2}$ for all $n\ge m$.

Also Σn^{-2} is convergent.

$\therefore$ By comparison test the given series is convergent.

Show that $\sum\frac{1}{4n^2-1}=\frac{1}{2}$

Solution.

Let $a_n=\frac{1}{4n^2-1}$

Clearly $a_n<\frac{1}{n^2}$ Also $\Sigma\frac{1}{n^2}$is convergent

$\therefore$ by comparison test, the given series converges

Now, $a_n=\frac{1}{4n^2-1}=\frac{1}{2}[\frac{1}{2n-1}-\frac{1}{2n+1}]$ (by partial fraction)

$\therefore s_n = a_1 + a_2 + \ldots + a_n$

$= \frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots.+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]$

$= \frac{1}{2}\left[\left(1-\frac{1}{2n+1}\right)\right]$

$\therefore \lim_{n\to\infty} s_n = \frac{1}{2}$

Hence $\sum \frac{1}{4n^2-1} = \frac{1}{2}$

Theorem 4.8 (Kummer's test)

Let Σa_n be a given series of positive terms and $\sum \frac{1}{d_n}$ be a series of a positive terms diverging to ∞. Then

(i) Σa_n converges if $\lim_{n\to\infty}\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) > 0$ and

(ii) Σa_n diverges if $\lim_{n\to\infty}\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) < 0$.

Proof.

(i) Let $\lim_{n\to\infty}\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) = l > 0$.

We distinguish two cases.

Case (i) l is finite.

Then given $\epsilon > 0$, there exists $m \in \mathbf{N}$ such that

$l - \epsilon < d_n \frac{a_n}{a_{n+1}} - d_{n+1} < l + \epsilon$ for all n ≥ m

$\therefore d_n a_n - d_{n+1} a_{n+1} > (l-\epsilon)\, a_{n+1}$ for all n ≥ m.

Taking $\epsilon = (1/2)l$, we get $d_n a_n - d_{n+1} a_{n+1} > (1/2) l a_{n+1}$ for all n ≥ m.

Now , let n ≥ m

$\therefore d_m a_m - d_{m+1} a_{m+1} > (1/2)\, l a_{m+1}$

$d_{m+1} a_{m+1} - d_{m+2} a_{m+2} > (1/2)\, l a_{m+2}$

…… …. ……. ………… ……………..

………………………………………………………

$d_{n-1} a_{n-1} - d_n a_n > (1/2)\, l a_n$

Adding, we get

$d_m a_m - d_n a_n > (1/2)\, l\, (a_{m+1} + \ldots + a_n)$

$d_m a_m - d_n a_n > (1/2)\, l\, (s_n - s_m)$ where $s_n = a_1 + a_2 + \ldots + a_n$
$d_m a_m > (1/2)\, l\, (s_n - s_m)$

$s_n < \frac{2d_m a_m + l s_m}{l}$ which is independent of n
∴ The sequence (s_n) of partial sums is bounded.
∴ a_n is convergent.
Case (ii) $l = \infty$.
Then given real number k > 0 there exists a positive integer m such that $d_n \frac{a_n}{a_{n+1}} - d_{n+1} > k$ for all n ≥m.
∴ $d_n a_n - d_{n+1} a_{n+1} > k a_{n+1}$ for all n ≥ m.
Now, let n ≥ m. Writing the above inequality for m, m+1,.....,(n – 1) and adding we get
$d_m a_m - d_n a_n > k\, (a_{m+1} + \cdots + a_n)$
$= k\, (s_n - s_m)$.
∴ $d_m a_m > k\, (s_n - s_m)$.
∴ $s_n < \frac{d_m a_m}{k} + s_m$
∴ The sequence (s_n) is bounded and hence Σa_n is convergent.

$(ii)\ \lim_{n\to\infty}\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) = l < 0$
Suppose l is finite.

Choose $\epsilon > 0$ such that $l + \epsilon < 0$. Then there exists $m \in$ **N** such that
$l + \epsilon < d_n \frac{a_n}{a_{n+1}} - d_{n+1} < l + \epsilon < 0$ for all n ≥ m.
∴ $d_n a_n < d_{n+1} a_{n+1}$ for all n ≥ m.
Now let n ≥ m
∴ $d_m a_m < d_{m+1} a_{m+1}$
....
....

$d_{n-1} a_{n-1} < d_n a_n$
∴ $d_m a_m < d_n a_n$.

∴ $a_n > \frac{d_m a_m}{d_n}$ Also by hypothesis $\sum \frac{1}{d_n}$ is divergent
Hence $\sum_{n=1}^{\infty} \frac{d_m a_m}{d_n}$ is divergent.
∴ By comparison test Σa_n is divergent.
The proof is similar if $l = -\infty$.

Corollary 1.(D' Alembert's ratio test)
Let Σa_n be a series of positive terms. Then Σa_n converges if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} > 1$ and diverges

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} < 1.$$

Proof.

The series 1 + 1 + 1 + ... is divergent

∴ We can put $d_n = 1$ in Kummer's test.

Then $d_n \frac{a_n}{a_{n+1}} - d_{n+1} = \frac{a_n}{a_{n+1}} - 1$

Hence Σa_n converges if $\lim_{n\to\infty}\left(\frac{a_n}{a_{n+1}} - 1\right) > 0$

Therefore Σa_n converges if $\lim_{n\to\infty}\frac{a_n}{a_{n+1}} > 1$

Similarly Σa_n diverges if $\lim_{n\to\infty}\frac{a_n}{a_{n+1}} < 1.$

Corollary 2. (Raabe's test)

Let Σa_n be a series of positive terms . Then Σa_n converges if $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1$ and diverges if $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) < 1.$

Proof. The series $\Sigma\frac{1}{n}$ is divergent.

∴ We can put $d_n = n$ in Kummer's test.

Then $d_n \frac{a_n}{a_{n+1}} - d_{n+1} = n\frac{a_n}{a_{n+1}} - (n+1)$

$= n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1$

∴ Σa_n converges if $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1$ and diverges if $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) < 1$

Theorem: 4.9 (Gauss's test)

Let Σa_n be a series of positive terms such that $\frac{a_n}{a_{n+1}} = 1 + \frac{\beta}{n} + \frac{r_n}{n^p}$ where p>1 and (r_n) is a bounded

sequence. Then the series Σa_n converges if $\beta > 1$ and diverges if $\beta \leq 1$.

Proof:

$\frac{a_n}{a_{n+1}} = 1 + \frac{\beta}{n} + \frac{r_n}{n^p}$, p>1

$n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{\beta}{n} + \frac{r_n}{n^p}\right) = \beta + \frac{r_n}{n^{p-1}}$

Now, since p>1, $\lim_{n\to\infty}\frac{1}{n^{p-1}} = 0$

Also (r_n) is a bounded sequence.

Hence $\lim_{n\to\infty}\frac{r_n}{n^{p-1}} = 0$

$\therefore \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \beta$

$\therefore$ By Raabes's test Σa_nconverges if $\beta > 1$ and Σa_ndiverges if $\beta < 1$.

If $\beta = 1$, Raabes's test fails. In this case we apply Kummer's test by taking $d_n = n \log n$

Now, $d_n \frac{a_n}{a_{n+1}} - d_{n+1} = n \log n \left(1 + \frac{1}{n} + \frac{r_n}{n^p}\right) - (n+1)\log(n+1)$

$= -(n+1)\log\left(1+\frac{1}{n}\right) + \frac{r_n \log n}{n^{p-1}}$

$= -\log\left(1+\frac{1}{n}\right)^{n+1} + \frac{r_n \log n}{n^{p-1}}$

Now, by hypothesis (r_n) is abounded sequence and $\left(\frac{\log n}{n^{p-1}}\right) \to 0$

$\therefore \left(\frac{r_n \log n}{n^{p-1}}\right) \to 0$

$\lim_{n\to\infty}\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) = -\log e = -1 < 0$

Hence by Kummer's test Σa_n diverges

Solved problems.

1. Test the convergence of the series$\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \cdots$ ...

Solution:

Let $a_n = \frac{1.2.3 \ldots\ldots n}{3.5.7\ldots.(2n+1)}$

$\frac{a_n}{a_{n+1}} = \frac{2n+3}{n+1} = \frac{2+\frac{3}{n}}{1+\frac{1}{n}}$

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 2 > 1$

Therefore by D' Alembert's ratio test Σa_n is convergent.

Theorem 4.10 (Cauchy's root test)

Let Σa_n be a series of positive terms. Then Σa_n is convergent if $\lim_{n\to\infty} a_n^{1/n} < 1$ and divergent if $\lim_{n\to\infty} a_n^{1/n} > 1$.

Proof.

Case(i) let $\lim_{n\to\infty} a_n^{\frac{1}{n}} = l < 1$.

Choose $\epsilon > 0$ such that $l + \epsilon < 1$.

Then there exists $m \in$ N such that $a_n^{1/n} < l + \epsilon$ for all $n \geq m$

$\therefore a_n < (l + \epsilon)^n$ for all $n \geq m$.

Now since $l + \epsilon < 1$, $\Sigma(l + \epsilon)^n$ is convergent.

$\therefore$ By comparison test Σa_n is convergent.

Case (ii) Let $\lim_{n\to\infty} a_n{}^{1/n} = l > 1$.

Choose $\epsilon > 0$ such that $l - \epsilon > 1$.

Then there exists m $\in$ **N** such that $a_n{}^{1/n} > l - \epsilon$ for all n ≥ m

∴ $a_n > (l - \epsilon)$ for all n ≥ m.

Now, since $l - \epsilon > 1$, $\Sigma (l - \epsilon)^n$ is divergent

∴ By comparison test, Σa_n is divergent.

Problems:

1. Test the convergence of $\sum \frac{1}{(\log n)^n}$

Solution:

Let $a_n = \frac{1}{(\log n)^n}$

$$\sqrt[n]{a_n} = \frac{1}{\log n}$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = 0 < 1$$

∴ by Cauchy's root test $\sum \frac{1}{(\log n)^n}$ converges.

2. Prove that the series $\sum e^{-\sqrt{n}} x^n$ converges if 0< x < 1 and diverges if x > 1.

Solution:

Let $a_n = e^{-\sqrt{n}} x^n$

$a_n{}^{1/n} = (e^{-\sqrt{n}} x^n)^{1/n}$

$\lim_{n\to\infty} a_n{}^{1/n} = x$

Hence by Cauchy's root test the given series converges if 0< x < 1 and diverges if x > 1.

UNIT - V
ALTERNATIVE SERIES

Definition: A series whose terms are alternatively positive and negative is called an alternating series.

Thus an alternating series is of the form

$a_1 - a_2 + a_3 - a_4 + ...= \sum (-1)^{n+1} a_n$ where $a_n > 0$ for all n.

For example

i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...= \sum (-1)^{n+1}\left(\frac{1}{n}\right)$ is an alternating series.

ii) $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + ...= \sum (-1)^{n+1}\left(\frac{n+1}{n}\right)$ is an alternating series.

We now prove a test for convergence of an alternating series.

Theorem :5.1(Leibnitz's test)

Let $\sum (-1)^{n+1} a_n$ be an alternating series whose terms an satisfy the following conditions

i) (a_n) is a monotonic decreasing sequence.

ii) $\lim_{n\to\infty} a_n = 0$.

Then the given alternating series converges.

Proof:

Let (s_n) denote the sequence of partial sums of the given series.

Then $s_{2n} = a_1 - a_2 + a_3 - a_4 + ... + a_{2n-1} - a_{2n}$

$s_{2n+2} = s_{2n} + a_{2n+1} - a_{2n+2}$

Therefore, $s_{2n+2} - s_{2n} = (a_{2n+1} - a_{2n+2}) \geq 0$ (by (i)).

Therefore, $s_{2n+2} \geq s_{2n}$.

Therefore, (s_{2n}) is a monotonic increasing sequence.

Also, $s_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - .... - (a_{2n-2} - a_{2n-1}) - a_{2n}$

$\leq a_1$ (by (i)).

Therefore, (s_{2n}) is bounded above.

Therefore, (s_{2n}) is a convergent sequence.

Let $(s_{2n}) \to s$.

Now, $s_{2n+1} = s_{2n} + a_{2n+1}$.

Therefore, $\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} a_{2n+1} = s + 0 = s$ (by (i))

Therefore, $(s_{2n+1}) \to s$.

Thus the subsequences (s_{2n}) and (s_{2n+1}) converges to the same limits.

Therefore, $(s_n) \to s$ (by theorem 3.29).

Therefore, The given series converges.

Problem : 1 Show that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$ converges.

Solution : The given series is $\sum (-1)^{n+1} a_n$ where $a_n = \frac{1}{n}$. Clearly $a_n > a_{n+1}$ for all n and hence (a_n) is monotonic decreasing.

Also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

∴ By Leibnitz's test the given series converges.

Problem : 2 Show that the series $\sum \frac{(-1)^{n+1}}{\log(n+1)}$ converges.

Solution : Let $a_n = \frac{1}{\log(n+1)}$.

Clearly $(a_n) \to 0$ as $n \to \infty$.

Also $\frac{1}{\log n} > \frac{1}{\log(n+1)}$ for all $n \geq 2$.

∴ By Leibnitz's test the given series converges.

Absolute convergence

Definition : A series $\sum a_n$ is said to be **absolutely convergent** if the series $\sum |a_n|$ is convergent.

Example : The series $\sum \frac{(-1)^n}{n^2}$ is absolutely convergent, for $\sum \left|\frac{(-1)^n}{n^2}\right| = \sum \frac{1}{n^2}$ which is convergent.

Theorem: 5.2

Any absolutely convergent series is convergent.

Proof :

Let $\sum a_n$ be absolutely convergent.

∴ $\sum |a_n|$ is convergent.

Let $s_n = a_1 + a_2 + a_3 + a_4 + \ldots\ldots\ldots\ldots + a_n$ and $t_n = |a_1| + |a_2| + \ldots\ldots\ldots\ldots + |a_n|$

By hypothesis (t_n) is convergent and hence is a Cauchy sequence

Hence given ε > 0, there exist $n_1 \in N$ such that $|t_n - t_m| < \varepsilon$ for all $n, m > n_1$ …………………… (1)

Now let $m > n$.

Then $|s_n - s_m| = |a_{n+1} + a_{n+2} + \ldots\ldots\ldots\ldots\ldots + a_m|$

$\leq |a_{n+1}| + |a_{n+2}| + \ldots\ldots\ldots\ldots + |a_m|$

$= |t_n - t_m| < \varepsilon$ for all $n, m > n_1$ (by (i)).

∴ (s_n) is a Cauchy sequence in R and hence is convergent

∴ $\sum a_n$ is a convergent series.

Definition : A series $\sum a_n$ is said to be conditionally convergent if it is convergent but not absolutely convergent.

Example : The series $\sum \frac{(-1)^n}{n^2}$ is conditionally convergent.

Theorem: 5.3

In a absolutely convergent series, the series formed by its positive terms alone is convergent and the series formed by its negative terms alone is convergent and conversely.

Proof :

Let $\sum a_n$ be the given absolutely convergent series.

We define $p_n = \begin{cases} a_n & if\ a_n > 0 \\ 0 & if\ a_n \leq 0 \end{cases}$ and $q_n = \begin{cases} 0 & if\ a_n \geq 0 \\ -a_n & if\ a_n < 0 \end{cases}$

(i.e) p_n is a positive terms of the given series and q_n is the modulus of a negative term.

$\sum p_n$ is the series formed with the positive terms of the given series and $q_n$is the series formed with the moduli of the negative terms of the given series.
Clearly $p_n \le |a_n|$and $q_n \le |a_n|$ for all n.
Since the given series is absolutely convergent, $\sum |a_n|$ is a convergent series of positive terms Hence by comparison test $\sum p_n$ and $\sum q_n$ are convergent.
Conversely $\sum p_n$ and $\sum q_n$ are converge to p and q respectively. We claim that $\sum a_n$ is absolutely convergent.
We have $|a_n| = p_n + q_n$
$\therefore \sum |a_n| = \sum (p_n + q_n)$
$= \sum p_n + \sum q_n$
$= p + q.$
$\therefore \sum a_n$ is absolutely convergent

Theorem: 5.4
If $\sum a_n$is an absolutely convergent series and (b_n) is a bounded sequence, then the series $\sum a_n b_n$ is an absolutely convergent series.
Proof :
since (b_n) is a bounded series , there exist a real number k > o such that $|b_n| \le k$ for all n.
$|a_n b_n| = |a_n||b_n|$
$\le k\ |a_n|$ for all n.
Since $\sum a_n$ is absolutely convergent $\sum |a_n|$ is convergent.
$\therefore \sum k\,|a_n|$ is convergent.
$\therefore$ By comparison test , $\sum |a_n b_n|$ is convergent.
$\therefore \sum a_n b_n$ is an absolutely convergent .

Problem 1 : Test the convergence of $\sum \frac{(-1)^n sinn\alpha}{n^3}$
Solution : We have $\left|\frac{(-1)^n sinn\alpha}{n^3}\right| \le \frac{1}{n^3}$ (since, $|sin\theta| \le 1$)
$\therefore$ By comparison test the series is a absolutely convergent.

Tests For Convergence of Series Of Arbitrary Terms
Theorem: 5.5
Let (a_n) be a bounded sequence and (b_n) be a monotonic decreasing bounded sequence. Then the series $\sum a_n(b_n - b_{n+1})$ is absolutely convergent.

Proof:
Since (a_n) and (b_n) are bounded sequences there exists a real number k> 0 such that $|a_n| \le$ k and
$|b_n| \le k$ for all n.

Let s_n denote the partial sum of the series $\sum |a_n(b_n - b_{n+1})|$

$$\therefore s_n = \sum_{r=1}^{n} |a_r(b_r - b_{r+1})$$

$= \sum_{r=1}^{n} |a_r|(b_r - b_{r+1})$

$\leq$ k $\sum_{r=1}^{n}(b_r - b_{r+1})$

= k $(b_1 - b_{n+1})$
$\leq$k (|b_1| +|b_{n+1}|)
$\leq$ k (k + k) =2k^2
$\therefore$ (s_n) is a bounded sequence.
$\therefore$ $\sum |a_n(b_n - b_{n+1})|$ is convergent.
Hence $\sum a_n(b_n - b_{n+1})$ is absolutely convergent.

Theorem: 5.6 (Dirichlet's test)
Let Σa_n be a series whose sequence of partial sums (s_n) is bounded. Let (b_n) be a monotonic decreasing sequence converging to 0. Then the series $\Sigma a_n b_n$ converges.

Proof:
Let t_n denote the partial sum of the series $\Sigma a_n b_n$

$$\therefore\ t_n = \sum_{r=1}^{n} a_r b_r$$

= s1 b1 + $\sum_{r=2}^{n}(s_r - s_{r-1})b_r$ (Since $s_r - s_{r-1} = a_r$)

= $\sum_{r=1}^{n-1}(b_r - b_{r+1})s_r + s_n b_n$................(1)

Since (s_n) is bounded and (b_n) is a monotonic decreasing bounded sequence $\sum_{r=1}^{n-1}(b_r - b_{r+1})s_r$

is a convergent sequence.
Also since (s_n) is bounded and $(b_n) \to 0, (s_n b_n) \to 0$
From (1) it follows that (t_n) is convergent.
Hence $\Sigma a_n b_n$ converges.

Theorem:5.7 (Abel's test)
Let Σ a_n be a convergent series. Let (b_n) be a bounded monotonic sequence. Then Σ $a_n b_n$ is convergent.
Proof:
Since (b_n) be a bounded monotonic sequence, $(b_n) \to b$(say)

$$\text{Let } c_n = \begin{cases} b - b_n \textit{ if } (b_n) \textit{ is monotonic increasing} \\ b_n - b \textit{ if } (b_n) \textit{ is monotonic decreasing} \end{cases}$$

$$\therefore\ a_n c_n = \begin{cases} a_n b - a_n b_n \textit{ if } (b_n) \textit{ is monotonic increasing} \\ a_n b_n - a_n b \textit{ if } (b_n) \textit{ is monotonic decreasing} \end{cases}$$

$$\therefore\ a_n b_n = \begin{cases} b a_n - a_n c_n \textit{ if } (b_n) \textit{ is monotonic increasing} \\ b a_n + a_n c_n \textit{ if } (b_n) \textit{ is monotonic decreasing} \end{cases} \text{.................(1)}$$

Clearly (c_n) is a monotonic decreasing sequence converging to 0. Also since Σ a_n is a convergent series its sequence of partial sums is bounded.
$\therefore$ by Dirichlet's test Σ $a_n c_n$ is convergent.

Also Σ a_n is convergent.

Σ b a_n is convergent.

Hence by (1) Σ a_nb_n is convergent.

Problems:

1. Show that convergence of Σa_n implies the convergence of $\sum \frac{a_n}{n}$

Solution:

Let Σa_n be convergent

The sequence (1/n) is a bounded monotonic sequence.

Hence by Abel's test $\sum \frac{a_n}{n}$ is convergent.2. Prove that $\sum_{n=2}^{\infty} \frac{\sin n}{\log n}$ is convergent.

Solution:

Let a_n = sin n and b_n= 1/log n.

Clearly (b_n) is a monotonic decreasing sequence converging to 0.

s_n = sin 2 + sin 3 +........ + sin (n+1)

$= \frac{1}{2} cosec \frac{1}{2} [\cos\left(\frac{3}{2}\right) - \cos\left(\frac{2n+1}{2}\right)]$

$\therefore |s_n| \le cosec(\frac{1}{2})$

(s_n) is a bounded sequence.

Hence by Dirichlet's test $\sum_{n=2}^{\infty} \frac{\sin n}{\log n}$ is convergent

Exercise:

1. Show that the series $\sum \frac{\sin n\theta}{n}$ converges for all values of θ and $\sum \frac{\cos n\theta}{n}$ converges if θ is not a
multiple of 2π.

MULTIPLICATION OF SERIES

Definition : Let ∑ a_n and ∑ b_n be two series.

Let $c_1 = a_1b_1$

$c_2 = a_1b_2 + a_2b_1$

$c_3 = a_1b_3 + a_2b_2 + a_3b_1$

....

....

....

$c_n = a_1bn + a_2b_{n-1} + a_3b_{n-2} +$ ....+ a_nb_1.

....

....

....

Then the series ∑ c_n is called the Cauchy product of ∑ a_n and ∑ b_n.

Example :

Consider the series $\sum \frac{(-1)^{n-1}}{(\sqrt{n})}$

We take the Cauchy product of the series with itself.

Let $a_n = \frac{(-1)^{n\,1}}{(\sqrt{n})} = b_n$.

Then $c_n = a_1 bn + a_2 b_{n-1} + a_3 b_{n-2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_n b_1$.

$= (-1)^{n-1}\left[\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{2}\sqrt{n-1}} + \frac{1}{\sqrt{3}\sqrt{n-2}} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{\sqrt{n}}\right]$

$\therefore |c_n| \geq \left[\frac{1}{\sqrt{n}\sqrt{n}} + \frac{1}{\sqrt{n}\sqrt{n}} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{\sqrt{n}\sqrt{n}}\right]$

$= n\,\frac{1}{n} = 1$.

$\therefore |c_n| \geq 1$ for all $n \in N$.

$\therefore$ The Cauchy product $\sum c_n$ is divergent.

However the given series $\sum \frac{(-1)^{n-1}}{(\sqrt{n})}$ converges (by Leibnitz's test).

Thus the Cauchy product of two convergent series need not converges.

Theorem: 5.8 (Abel's theorem).

If $\sum a_n$ and $\sum b_n$ converge to a and b respectively and if the Cauchy product $\sum c_n$ converges to c, then c = ab.

Proof:

Let $A_n = a_1+a_2+\ldots\ldots\ldots\ldots + a_n$.

$B_n = b_1+b_2+\ldots\ldots\ldots\ldots + b_n$.

$C_n = c_1+c_2+\ldots\ldots\ldots\ldots + c_n$.

$\therefore\ C_n = a_1b_1 + (a_1b_2 + a_2b_1) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + (a_1b_n + a_2b_{n-1} + \ldots\ldots\ldots + a_nb_1)$

$= a_1(b_1 + b_2 + \ldots\ldots\ldots + b_n) + a_2(b_1 + b_2 + \ldots\ldots\ldots + b_{n-1}) + \ldots\ldots\ldots\ldots\ldots\ldots + a_nb_1$

$= a_1B_n + a_2B_{n-1} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_nB_1$ -- (1)

From (1) $C_1 = a_1B_1$

$C_2 = a_1B_1 + a_2B_1$

……………………

……………………

$C_n = a_1B_n + a_2B_{n-1} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_nB_1$

$\therefore\ C_1 + C_2 + \ldots\ldots\ldots + C_n$

$= a_1B_1 + (a_1B_1 + a_2B_1) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + (a_1B_1 + a_2B_2 + \ldots\ldots\ldots\ldots + a_nB_n)$

$= B_1(a_1+a_2+\ldots\ldots\ldots\ldots + a_n) + B_2(a_1+a_2+\ldots\ldots\ldots\ldots + a_{n-1}) + \ldots\ldots\ldots\ldots + B_na_1$

$= A_n B_1 + A_{n-1}B_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + A_1 B_n$.

By hypothesis $\sum a_n$ converges to a and $\sum b_n$ converges to b.

$\therefore\ (A_n) \to a$ and $(B_n) \to b$.

Hence by Cesaro's theorem,

$\left(\frac{A_1B_n + A_2B_{n-1} + \cdots\ldots\ldots\ldots\ldots\ldots\ldots + A_nB_1}{n}\right) \to$ ab.

i.e., $\left(\frac{C_1 + C_2 + \cdots\ldots\ldots\ldots\ldots\ldots\ldots + C_n}{n}\right) \to$ ab.

Also by hypothesis $\sum c_n$ converges to c

∴ (Cn) $\to$ c.

Hence by Cauchy's first limit theorem,

$$\left(\frac{c_1 + c_2 + \cdots\cdots\cdots + c_n}{n}\right) \to c$$

∴ $c = ab$.

Theorem 5.9 (Merten's Theorem)

If the series $\sum a_n$ and $\sum b_n$ converge to the sums a and b respectively and if one of the series, say, $\sum a_n$ is absolutely convergent, then the Cauchy product $\sum C_n$ converges to the sum ab.

Proof :

Let $A_n = a_1+a_2+\ldots\ldots\ldots\ldots + a_n$.

$B_n = b_1+b_2+\ldots\ldots\ldots\ldots + b_n$.

$C_n = c_1+c_2+\ldots\ldots\ldots\ldots +c_n$.

$\overline{A_n} = |a_1| + \ldots\ldots\ldots\ldots + |a_n|$

and $\sum |a_n| = \bar{a}$, so that $(\overline{A_n}) \to \bar{a}$.

Now , let $B_n = b + r_n$.

Since, $(B_n) \to b$, $(r_n) \to 0$ as $n \to \infty$.

Now , $C_n = a_1B_n + a_2B_{n-1} + \ldots\ldots\ldots\ldots + a_nB_1$

$= a_1(b + r_n) + a_2(b + r_{n-1}) + \ldots\ldots\ldots\ldots + a_n(b + r_1)$

$= (a_1 + \ldots\ldots\ldots + a_n)b + (a_1r_n + \ldots\ldots\ldots + a_nr_1)$

$= A_n b + (a_1r_n + \ldots\ldots\ldots + a_nr_1)$

$= A_n b + R_n$ where $R_n = a_1r_n + \ldots\ldots\ldots + a_nr_1$.

Since, $(A_n) \to a$, $(A_n b) \to ab$.

∴ To prove that $(C_n) \to a b$, it is enough if we prove that $(R_n) \to 0$

Let $\varepsilon > 0$ be given. Since $(r_n) \to 0$, there exist $n_1 \in N$ such that $|r_n| < \varepsilon$ for all $n \geq n_1$. ………………(1) Also since the sequence (r_n) is convergent, it is a bounded sequences and hence there exists $k \geq 0$ such that $|r_n| < k$ for all n. …………………………… (2)

Further since $(\overline{A_n}) \to \bar{a}$, $(\overline{A_n})$ is a Cauchy sequence.

∴ There exists $n_2 \in N$ such that $|\overline{A_n} - \overline{A_m}| < \varepsilon$ for all $n, m \geq n_2$……………………(3)

Let $p = \max\{n_1, n_2\}$,

Let $n \geq 2p$.

Then $R_n = a_1r_n + a_2r_{n-1} + \ldots\ldots\ldots + a_pr_{n-p+1} + a_{p+1}r_{n-p} + \ldots\ldots + a_nr_1$.

$$\therefore |R_n| \leq \{|a_1||r_n| + |a_2||r_{n-1}| + \ldots\ldots\ldots + |a_p||r_{n-p+1}|\} + \{|a_{p+1}||r_{n-p}| + \ldots\ldots\ldots + |a_n||r_1|\}$$

Now $n \geq 2p \Rightarrow n, n-1, \ldots\ldots, (n - p - 1) \geq p \geq n_1$.

∴ $|a_1||r_n| + |a_2||r_{n-1}| + \ldots\ldots\ldots + |a_p||r_{n-p+1}|$

$< (|a_1| + |a_2| + \ldots\ldots\ldots + |a_p|)\,\varepsilon$ (by 1).

= $\overline{A_p}\varepsilon$

$<\bar{a}\varepsilon$ (since $(\overline{A_p})$ is a monotonic increasing sequence converging to $\bar{a}$) ………………… (5)

Also, $|a_{p+1}||r_{n-p}|$ + ……….+$|a_n||r_1|$

$\leq$ $(|a_{p+1}| + |a_{p+2}| + \dots\dots\dots\dots\dots\dots + |a_n|$ k (by 2)

$\leq$ $(\overline{A_n}$ - $\overline{A_p})$ k

$<\varepsilon\, k$ $(\, by\, 3)$

∴ Using (5) and (6) in (4) we get

$|R_n| < (\bar{a} + k)\varepsilon$ for all n $\geq$ 2p.

∴ $(R_n) \to 0$.

∴ (c_n) converges to a b.

∴ $\sum C_n$ converges to a b.

Power Series

Definition:

A series of the form $a_0 + a_1x + a_2x^2 + \cdots .. + a_nx^n + \cdots .. = \sum_{n=0}^{\infty} a_nx^n$ is called a power series in x. The number a_n are called the coefficients of the power series.

Example:

Consider the geometric series $\sum_{n=0}^{\infty} x^n$.Here a_n =1 for all n. This series converges absolutely if $|x| < 1$,

diverges if $x \geq 1$, oscillates finitely if x = -1 and oscillates infinitely if x < -1

Theorem: 5.10

Let $\sum a_nx^n$ be the given power series. Let α = $\lim sup\, |a_n|^{\frac{1}{n}}$ and let $R = \frac{1}{\alpha}$ .Then $\sum a_nx^n$ converges absolutely if $|x| < R$. If $|x| > R$ the series is not convergent.

Proof:

Let $c_n = a_nx^n$

∴ $|c_n|^{\frac{1}{n}} = |c_n|^{\frac{1}{n}}|x|$

∴ $lim\, sup\, |c_n|^{\frac{1}{n}} = |x|\; lim\, sup\, |a_n|^{\frac{1}{n}}$

$= |x|\frac{1}{R}$

Hence By Cauchy's root test the series converges if $\frac{|x|}{R} < 1$.

i.e) if $|x| < R$

Now suppose $|x| > R$ .Choose a real number μ such that $|x| > \mu > R$.

∴ $\frac{1}{\mu} < \frac{1}{R} = lim\, sup\, |a_n|^{\frac{1}{n}}$

Hence by definition of upper limit, for infinite number of values of n we have

$|a_n|^{\frac{1}{n}} > \frac{1}{\mu} > \frac{1}{|x|}$

$\therefore |a_n x^n| > 1$ for finite number of values of n.
Hence the series cannot converge.

Definition:
The number $R = \frac{1}{\lim\sup |a_n|^{\frac{1}{n}}}$ given in the above theorem is called the radius of convergence of the power series $\Sigma a_n x^n$

Example:
1. For the geometric series $\sum x^n$,the radius of convergence R =1
2. Consider the exponential series $1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots . + \frac{x^n}{n!} + \cdots$
Here $a_n = \frac{1}{n!}$

$$\left|\frac{a_n}{a_{n+1}}\right| = n + 1$$

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \infty.$$

$\therefore R = \infty.$
Hence the series converges for all values of x.