UNIT I

RELATIONS

1.1 INTRODUCTION TO RELATION

In Mathematics, the expressions such as 'is less than', 'is parallel to', 'is perpendicular to' are relations.

Relations may exist between objects of the same set or between objects of two or more sets.

1.2BINARY RELATION

Let A and B be two non-empty sets.

Then any subset of R of the Cartesian product AxB is called a binary relation R from A to B.

If $(a,b)\epsilon R$, then a is related to b and is written as aRb.

The set $\{a \in A: (a,b)\epsilon R$ for some b∈B$\}$ is called the **domain** of R and is denoted by $D_R(R)$.

The set $\{b \in B: (a,b)\epsilon R$ for some a∈A$\}$ is called the **range** of R and is denoted by $R_R(R)$.

Example 1.1

i. Let $A = \{3,6,9\}, B = \{4,8,12\}$.

 Then $R = \{(3,4),(3,8),(3,12)\}$ is a relation from A to B

ii. Let $A = \{2,3,4\},\ B = \{a,b\}$.

 Then AxB= $\{(2,a),(2,b),(3,a),(3,b),(4,a),(4,b)\}$

 If R =$\{(2,a),(3,b)\}$, then R $\subseteq$ A xB and

 R is a relation from A to B.

iii. Let $A = \{2,3,4\},\ and\ B = \{3,4,5,6,7\}$.

If R is a relation from A to B defined by

(a,b)ϵ R such that a divides b (with zero remainder) then,

R={(2,4), (2,6), (3,3), (3,6), (4,4)}

Example 1.2:

Let$A = \{1,2,3,4\}$ and $B = \{3,4,5,6\}$

Find the elements of each relation R stated below. Also, find the domain and range of R.

i. aRb if and only if a<b
ii. aRb if and only if a and b are both odd numbers.

Solution:

i. $R = \{(1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6)\}$

$Domain\ of\ R = D_R(R) \quad = \{1,2,3,4\}\ and\ Range\ of\ R = R_R(R) \quad = \{3,4,5,6\}$

ii. $R = \{(1,3), (1,5), (3,3), (3,5)\} R_R(R) \quad = \{1,3\}\ and R_R(R) \quad = \{3,5\}$

Example 1.3:

Let A = {1,2,9} and B={1,3,7}
Find the elements of each relation R stated below. Also, find the domain and range of R if

(i) relation R is 'equal to' then
(ii) relation R is 'less than'
(iii) relation R is 'greater than'

Solution:
(i) R={(1,1),(3,3)}
Dom(R) = {1,3}
Ran(R) = {1,3}

(ii) R={(1,3),(1,7),(2,3),(2,7)}

Dom(R) = {1,2}

Ran(R) = {3,7}

(iii) R={(2,1),(9,1),(9,3),(9,7)}

Dom(R) = {2,9}

Ran(R) = {1,3,7}

Example 1.4:

Find the number of distinct reactions from a set A to Set B

Solution:

Let the number of elements in A and Bbe m and n respectively.

A x B has mn elements.

∴ Power set of A x B has 2^{mn} elements.

(i.e.,) A x B has 2^{mn}distinct Subsets.

Every subset of A x B is a relation from A to B.

Thus the number of distinct relations from A to B is 2^{mn}.

Note:

Let R be a relation defined on a set A consisting of n elements.

A x A contains 2^{n2} elements.

∴ There exists 2^{n2} binary relations on a set A.

1.3 CLASSIFICATION OF RELATIONS

In several applications of computer science and applied mathematics, we generally treat relations on a set A rather than relations from A to B. Furthermore, these relations often satisfy certain properties. The various types of relations are explained in this section.

Reflexive Relation

The relation Rdefined on a Set A is said to be reflexive if aRa (or (a,a) ϵR) for all aϵA.

Example:

i. Let R be a relation on $A = \{1,2,3,4\}$

   Then the relation R = $\{(1,1),(2,2),(3,3),(4,4)\}$ is reflexive.

ii. Let A=$\{a,b,c\}$ and $R = \{(a,a),(b,b),(c,c)\}$

   Then R is a reflexive relation on A.

Symmetric Relation:

A relation R defined on a set A is said to be symmetric ifaRb ⇒ bRa for all a, b ϵA.

i.e., R is symmetric onA if(a,b) $\epsilon R \Rightarrow (b,a) \epsilon R$.

Example:

(i) Let A={1,2,3} and R={(2,2),(2,3),(3,2)}.
Then R is symmetric, since both(2,3) ϵ R and (3,2) ϵ R.

(ii) Let R be a relation defined by ‘is perpendicular to’ on the set of all straight lines.
If line a is perpendicular to b, then b is perpendicular to a.
Then R is a symmetric relation.

Antisymmetric Relation

A relation R defined on a set A is said to be antisymmetricif(a,b) ∈R and a≠b,then (b,a) ∉R for all a,b, ∈A

Example:

Let R be a relation defined on A={1,2,3} by (a,b)∈R if a ≤b, for a,b∈A.

Then R = {(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)}.

Here, (1,2) ∈ R, but (2,1) ∉ R.

∴The relation R is antisymmetric.

Transitive Relation

A relation R defined on a set A is said to be transitive, if

(a,b) ∈R and (b,c) ∈ R ⇒(a,c)∈ R for all a,b,c∈ A

i.e., aRb and bRc ⇒aRc for all a,b,c∈A.

Example:

i. Let $A = \{1,2,3\}$ and $R = \{(1,1),(2,2),(2,3),(3,2),(3,3)\}$.

   Then R is transitive, since 2R2 and $2R3 \Rightarrow 2R3$

   also, 2R3and 3R2 ⇒ $2R2$.

ii. Let R be a relation on A=$\{a,b,c,d\}$given by

   $R = \{(a,a),(b,c),(c,b),(d,d)\}$

   Here (b,c) $\epsilon R$and (c,b) ϵRbut (b,b) ∉ R.

   So, the relation R is not transitive.

iii. Let A denote the set of straight lines in a plane and R be a relation onA defined by “is parallel to”.

   Let a, b, c be three lines. If a is parallel to b and b is parallel to c, then ais parallel to c. Hence R is a transitive relation on A.

Equivalence Relation

A relation R on a set A is said to be an equivalence relation, if R is reflexive, symmetric and transitive.

Example:

Let R be a relation defined on A=$\{a,b,c\}$by

$R = \{(a,a),(a,b),(b,a),(b,b)(b,c),(a,c),(c,a),(c,b),(c,c)\}$

Then R is an equivalence relation.

Associative Relation

Relations from A to B are subsets of A x B. Two relations from A to B can also be associated in the same way as two sets can be associated.

Let A $= \{a, b, c\}$ and B $= \{a, b, c, d\}$

Let $R_1 = \{(a,a),(b,b),(c,c)\}$ and $R_2 = \{(a,a),(a,b),(a,c),(a,d)\}$

The associative relations of A and B are

$R_U = R_1 \cup R_2 = \{(a,a),(b,b),(c,c),(a,b),(a,c),(a,d)\}$

$R_\cap = R_1 \cap R_2 = \{(a,a)\}$

$R_{12} = R_1 - R_2 = \{(b,b),(c,c)\}$

$R_{21} = R_2 - R_1 = \{(a,b),(a,c),(a,d)\}$

Example 1.5:

Let A= $\{1,2,3\}$. Check whether the following relations are reflexive, symmetric, anti symmetric or transitive.

i. R= $\{(1,1), (2,2), (3,3), (1,3), (1,2)\}$
ii. R= $\{(1,1), (2,2), (1,3), (3,1)\}$
iii. R= $\{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2)\}$

Solution:

i. The given relation R is reflexive and transitive.
   R is not symmetric Since $(1,3) \in$ R but $(3,1) \notin$ Rand
   $(1,2) \in R$ but $(2,1) \notin R$
ii. The relation R is symmetric.
   R is not reflexive, since $(3,3) \notin R$.
   R is not transitive since $(3,1) \in$ R and $(1,3) \in$ R but $(3,3) \notin R$.
iii. The given relation R is reflexive and symmetric.
   R is not transitive since $(1,2) \in$ R and $(2,3) \in$ R but $(1,3) \notin R$.

Example 1.6:

Let Z^* be the set of all non-zero integers and R be the relation on Z^* such that (a,b)ЄR if a is the factor of b i.e., a/b. Investigate R for reflexive, symmetric, anti symmetric or transitive.

Solution:

(i) $\frac{a}{a}\ \forall a \in Z*$

∴ R is reflexive.

(ii) a/b does not imply b/a

∴ R is not symmetric.

(iii) If 4/4 and -4/4 are true then $4\neq-4$

∴ R is not anti symmetric.

(iv) If a/b and b/c then a/c.

∴ R is transitive.

Example 1.7:

Let Z denote the set of integers and the relation R in Z be defined by aRbiff a-b is an even integer.Then show that R is an equivalence relation.

Solution:

(i) a-a=0 which is an even integer

aRa$\forall a \in z$

∴ R is reflexive.

(ii) Let aRb

$\Rightarrow$ a - b is a even integer

$\Rightarrow$ $-$(a-b) is an even integer

$\Rightarrow$ $b -$a is an even integer

∴bRa

i.e,, aRb$\Rightarrow bRa$

∴ R is symmetric.

(iii) Let aRb and bRc

aRb$\Rightarrow$a-b is an even integer

bRc$\Rightarrow$b-c is an even integer.

a-c = (a-b)+(b-c)

∴(a-c) is an even integer.

i.e aRc

aRb and bRc$\Rightarrow$aRc

∴R is transitive

R is reflexive symmetric and transitive.

Thus R is an equivalence relation.

Example 1.8:

Let A be the set of all triangles in the Euclidean plane and R is the relation on A definedby ‘a is similar to b’. Then show that R is an equivalence relation on A.

Solution:

i. Every triangle is similar to itself.

i.e, the relation R is reflexive.

ii. If a is similar to b then b is similar to a.

i.e$aRb \Rightarrow bRa$

$\therefore$ Ris Symmtric.

iii. If a is similar to b and b is similar to c, then a is similar to c.

$\therefore$ R is transitive.

R is reflexive, symmetric and transitive.

$\therefore$ R is an equivalence reaction

1.4 COMPOSITION OF RELATION

Let R_1 be a relation from A to B and R_2 be a relation from B to C.

The composition of R_1 and R_2 denoted by $R_2 o R_1$ is the relation from A to C defined as

$R_2 o R_1 = \{(a,c): (a,b) \epsilon R_1 and (b,c) \epsilon R_2$ for some $b \epsilon B\}$

Example 1.9:

Find the composition of the Relations.

$R_1 = \{(1,2), (1,6), (2,4), (3,4), (3,6), (3,8)\}$ and

$R_2 = \{(2,x), (4,y), (4,z), (6,z), (8,x)\}$

Solution:

$R_2 o R_1 = \{(1,x), (1,z), (2,y), (2,z), (3,y), (3,z), (3,x)\}$

1.5 INVERSE OF RELATION

Let R be a relation from A to B. The inverse of R is denoted by R^{-1}, and it is a relation from B to A defined by

$R^{-1} = \{(b,a): (a,b) \epsilon R\}$

Example 1.10:

Let A= {2,3,4}, B= {3,4,5,6,7}, and

R= {(2,5), (2,6), (3,3), (3,7), (4,4)}. Find the inverse of R.

Solution:

The inverse of the relation R is

$$R^{-1} = \{(5,2), (6,2), (3,3), (7,3), (4,4)\}$$

Example 1.11:

If a relation R is transitive, then prove that its inverse relation R^{-1} is also transitive.

Solution:

Let (a,b) ϵR and (b,c) ϵR^{-1}.

⇒ (b,a) ϵR and (c,b) ϵR

⇒ (c,b) ϵR and (b,a) ϵR

⇒ (c,a) ϵR [Since R is transitive]

⇒ (a,c) ϵR^{-1}

∴ R^{-1} is transitive.

1.6 REPRESENTATION OF RELATIONS ON A SET

A relation on a set A is a relation from A to A. i.e., a relation on a set A can be treated as a subset of A x A.

Example 1.12:

Let A = {1,2,3,4}. Let R = {(a,b) : a divides b}. Find the ordered pairs which exists in R.

Solution:

R = {(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)}

1.7 CLOSURE OPERATIONS ON RELATIONS

Let R be relation on a set A. The relation R may or may not possess the relational properties such as reflexivity, symmetry, and transitivity. If R does not possess any property, to fulfill R with a property, we should add new pairs to R. The smallest relation R1 on A which contains R and possesses the required property is called the closure of the relation R.

1.17.1 Reflexive Closure

Let R be a relation defined on a set A.

Relation R_R is called *reflexive closure* of *R* if R_R is the smallest relation containing *R*, having the reflexive property.

i.e., $R_R = R \cup \Delta_A$ where $\Delta_A = \{(a, a): a \in A\}$ is the *diagonal* or *equality* relation on A.

In other words, the reflexive closure of *R* can be obtained by adding to *R* all pairs of the form (a, *a)*, *a*∈A, not already in R.

Example 1.13:

Find the reflexive closure of the relation R = {(1, 1), (1, 2), (2, 1), (3, 2)} on the set A = {1,2,3}.

Solution:

This relation is not reflexive.

The relation can be made reflexive, by adding (2, 2) and (3, 3) to *R.*

Hence the reflexive closure of R is {(1,1), (2,2), (3,3),(1,2), (2,1), (3,2)}

Example 1.14:

Find the reflexive closure of the relation R = {(a, a), (a, b), (b, c), (c, a)} on the set A = {a,b,c}.

Solution:

The relation can be made reflexive, by adding (b, b) and (c, c) to *R*. Hence, the reflexive closure of R is

$$R_R = R \cup \Delta_A = \{(a, a), (b, b), (c, c), (a, b), (b, c), (c, a)\}$$

1.17.2 Symmetric Closure

Let R be a relation defined on a set A.

The relation *Rs* is called the *symmetric closure* of *R* if R_S is thesmallest relation containing *R*, having the symmetric property.

The relation $R_S = R \cup R^{-1}$ is thesmallest symmetric relation containing *R*and it is the symmetric closure of *R*.

Inotherwords, the symmetric closure of *R* is obtained by adding all ordered pairs of the form *(b, a)*, whenever *(a, b)*, belongs to the relation, that are not already present in *R*.

Example 1.15:

Find the Symmetric Closure of the relation

$$R = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2)\}$$

Defined on the set A = {1,2,3}.

Solution:

The relation is not symmetric.

The relation will be a symmetric if we add (2,1) and (1,3) to *R*. Hence the symmetric closure of R is

{(1,1), (2,1), (2,2), (2,3), (3,1), (1,3), (3,2)}

Example 1.16:

Find the symmetric closure of the relation

R = {(4, 5), (5, 5), (5, 6), (6, 7), (7, 4), (7, 7)}

defined on the set A = {4,5,6,7}.

Solution*:*

The smallest relation containing *R*, having the symmetric property, is

$R\cup R^{-1}$={(4,5),(5,4),(5,5),(5,6),(6,5),(6,7),(7,6),(7,4),(4,7),(7,7)}

1.7. MATRIX REPRESENTATION OF RELATION

Let A = $\{a_1, a_2, \dots \dots, a_m$ and B = $\{b_1, b_2, \dots b_n\}$be two finite sets, containing m and *n* elements, respectively.
Let R be a relation from A to B.
Then, *relation matrix* of *R*, denoted by M_R is an *m*x*n* matrix, i.e.,

$$M_{R=}[m_{ij}]_{m\ Xn}$$

where

$$m_{ij} = \begin{cases} 0, & if\ (a_i, b_j) \notin R \\ 1, & if(a_1, b_j) \in R \end{cases}$$

M_R can be described both in the tabular and in the matrix form.

Example 1.17:

Let A= {1,2,3} and R={(1,2), (1,3), (2,3)}.Determine M_R.

Solution:

	1	2	3
1	0	1	1
2	0	0	1
3	0	0	0

$$M_R=\begin{bmatrix}0 & 1 & 1\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

Example 1.18 :

Let A = {1,2, 3}, B = {*a*, *b*), *and* R = {(1, a), (2, *b*), (3, a)}. Determine M_R in tabular form and in matrix forms.

Solution:

	a	b
1	1	0
2	0	1
3	1	0

$$M_R=\begin{bmatrix}1 & 0\\0 & 1\\1 & 0\end{bmatrix}$$

Example 1.19:

Let A = {1, 4, 5} and R = {(1, 4), (1, 5), (4, 1), (4, 4), (5, 5)}. Determine M_R.

Solution*:*

Given that R = {(1, 4), (1, 5), (4, 1), (4, 4), (5, 5)}.

Then the relation matrix of R, given by

$$M_R=\begin{bmatrix}0 & 1 & 1\\1 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Example 1.20:

Let A = {1, 2, 3,4}, B = {p, q, r,5}, and

R = {(1, p), (1, q), (1, r), (2, q), (2, r), (2, i)}. Find M_R.

Solution:

The matrix representation of relation R, i.e., M_R is given by

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} p & q & r & s \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example 1.21 :

A= {a,b,c} and $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Find the relation R defined on A.

Solution:

M_R can be re-written as

$$M_R = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Thus, R= {(a,a), (a,b), (b,c)}

1.8 DIGRAPHS

Let R be a relation defined on the set A = {$a_1, a_2, \ldots, a_n$}.

The element a_i of A are represented by points or circles called *nodes* or *vertices.*

If $(a_i, a_j) \epsilon R$, then we connect the vertices a_i and aj by means of an arc and place an arrow in the direction from a_i to a_j.

If $(ai, aj) \epsilon R$ and $(aj, ai) \epsilon R$, then we draw two arcs one between a_i to a_j and the other between a_j to a_i.

When all the nodes corresponding to the ordered pairs in R are connected by arcs with proper arrows, we get a graph of the relation R.

This diagram or graph is called the *directed graph*, or *digraph* of the relation R.

If R is a relation on a set A, a **path** of *lengthn* in R from ai to aj is a finite sequence P, such as, $a_i, a_1, a_2, \ldots, a_{n-1}, a_j$ beginning with a_i, and ending with aj such that $a_i R\, a_1, a_1 R\, a_2, \ldots, a_{n-1} R\, a_j$

If n is a positive integer then the relation R^n on the set A can be defined as that there is a path of length n from a_i to aj in R *i.e.*, $(a_i, a_j) \in R^n$.

The relation R^∞ is defined on A or $(a_i, a_j) \in R^\infty$ means, that there is some path in R from a_i to aj.

A path that begins and ends at the same vertex is called a *cycle.*

A *cycle* in a digraph can be defined as a path of length $n \geq 1$ from a vertex to itself.

The relation R^∞ is sometimes called the *connectivity relation* for R.

The $R^n(x)$ consists of all vertices that can be reached from x by means of a path in R of length n.

The set $R^\infty(x)$ consists of all vertices that can be reached from x by some path in R.

If R is reflexive, then there must exists a *loop* at each node in the digraph of R.

If R is symmetric, then $(a_i, a_j) \in R$ implies $(a_j, a_i) \in R$ and the nodes a_i and a_j will be connected by two arcs (edges), i.e., one from a_i to a_j and the other from aj to a_i .

Example 1.22:

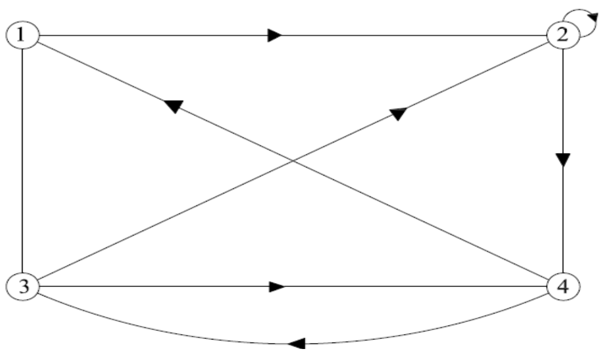
Draw the directed graph or digraph of the relation

R = {(1, 2), (1,3), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)}

on the set A = {1,2, 3, 4}.

Solution:

The digraph of R is



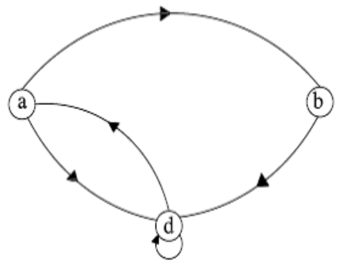
Example 1.23:

Let ***A*** = ***{a, b, d)*** and ***R* =*{(a, b), (a,d), (b, d), (d, a), (d, d)}*** be a relation on ***A.***

Draw the digraph of R.

Solution:

The digraph of *R* is shown in Figure

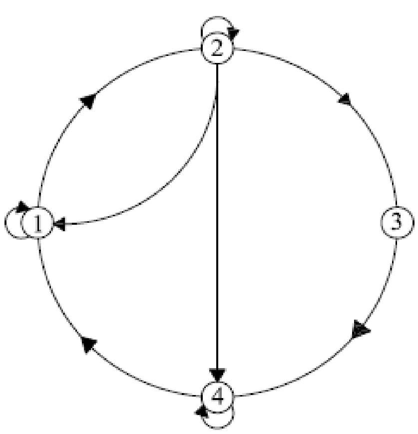


Example 1.24:

Let A = {1,2, 3, 4} and *R* = {(1,1), (1, 2), (2,1), (2,2), (2,3), (2,4), (3,4), (4,1), (4,4)}.Construct the digraph of *R.*

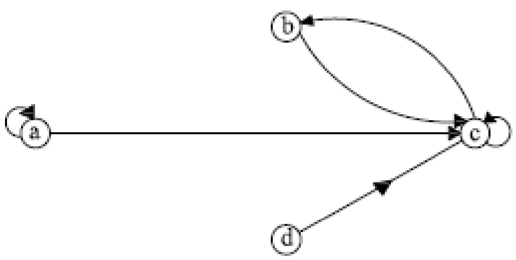
Solution:

The digraph of *Ris*



Example 1.25:

Find the relation R from the digraph of the following figure.



Solution :

The relation R for the digraph is R = {(a,a),(a,c),(b,c),(c,b),(c,c)(d,c)}

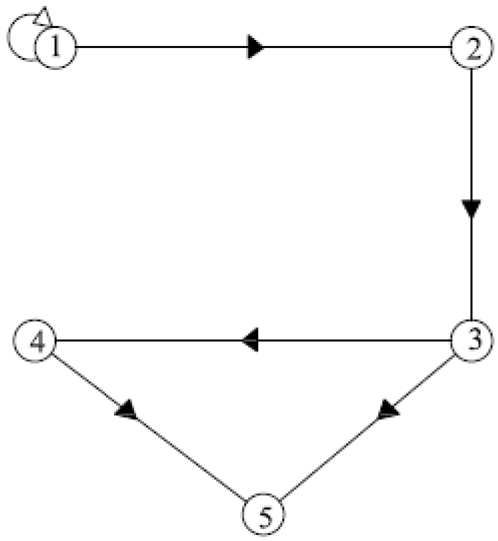
Example 1.26:

Let A = {1,2, 3,4,5} and R = {(1,1), (1,2), (2,3), (3,5), (3,4), (4,5)}.

Determine (i) R^2 (ii) R^∞

Solution:

The digraph of R is



(i) Here, (1, l)∈ R and(l, 1) ∈R⇒(1, 1) ∈R^2Again,

$$(1,1)\epsilon R \ and \ (1,2)\epsilon R \Rightarrow (1,2)\epsilon R^2$$

$$(1,2)\epsilon R \ and \ (2,3)\epsilon R \Rightarrow (1,3)\epsilon R^2$$

$$(2,3)\epsilon R \ and \ (3,5)\epsilon R \Rightarrow (2,5)\epsilon R^2$$

$$(2,3)\epsilon R \ and \ (3,4)\epsilon R \Rightarrow (2,4)\epsilon R^2$$

$$(3,4)\epsilon R \ and \ (4,5)\epsilon R \Rightarrow (3,5)\epsilon R^2$$

Thus,

R^2 = {(1, 1), (1, 2), (1, 3), (2, 5), (2, 4), (3, 5)}

(ii)

R^∞ = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)}

Example 1.27:

Find a non-empty set and a relation on the set that satisfy each of the following combinations of properties. Simultaneously, draw a digraph of each relation.

(i) Reflexive and symmetric but not transitive.

(ii) Reflexive and transitive but not antisymmetric.

Solution:

(i) Let A = {a, b, c} and

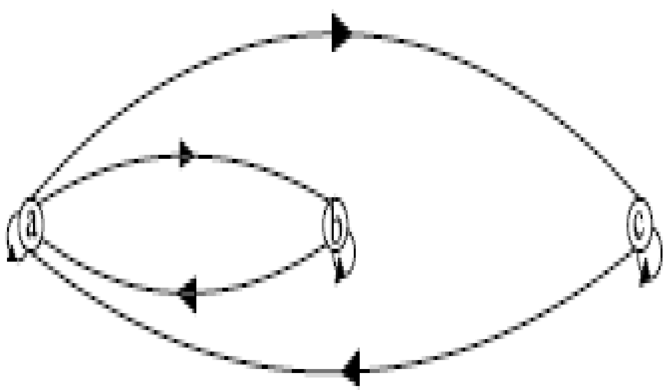
R = {(a, a), (b, b), (c, c), (a, b), (a, c), (b, a), (c, a)}.

R is reflexive, since for each element a∈A, (a, a) ∈R.

R is symmetric, since both (a, b) and (b, a) are in R.

Also (a,c) and (c,a) are in R.

R is not transitive, because (b, a) and (a, c) are in R, but (b, c) ∉ R.



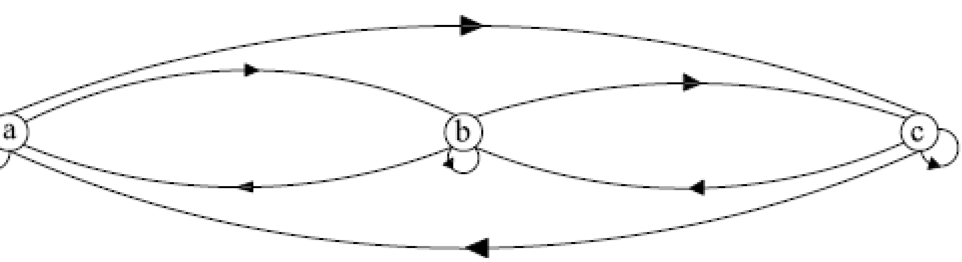
(ii) Let A = {a, b, c} and

R = {(a, a), (b, b), (c, c), (a, b), (a, c), (b, c), (b, a), (c, b), (c, a)}.

R is reflexive.

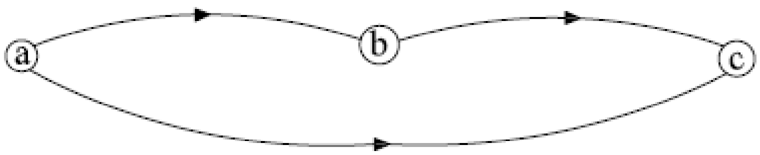
R is transitive, since aRb and bRc implies aRc.

R is not antisymmetric because (a, b) ∈R and a≠b, then (b, a) ∈R



Example 1.28:

Find the relation R for the digraph in the following figure.



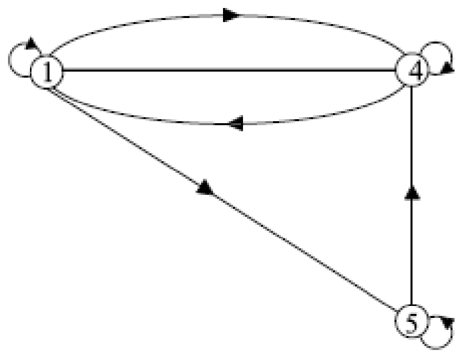
Solution:

R = {(a,b),(b,c),(a,c)}.

R is transitive and antisymmetric.

Example 1.29:

Let R be a relation defined on A = {1,4,5}. The digraph of R is shown in the Figure.



Determine M_R and R.

Solution:

Here R={(a,b),(b,c), (a,c)}

$$M_R = \begin{matrix} & 1 & 4 & 5 \\ 1 & 1 & 1 & 1 \\ 4 & 1 & 1 & 0 \\ 5 & 0 & 1 & 1 \end{matrix}$$

Example 1.30:

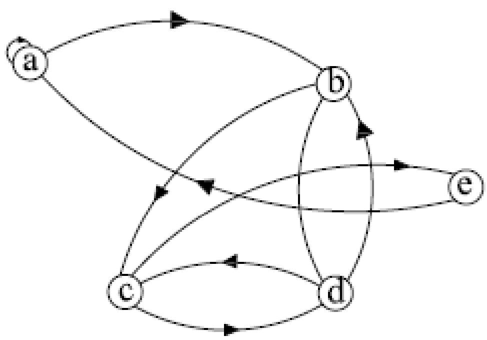
Let $A = \{a,b,c,d,e\}$ *and* $M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

Find the relation *R* defined on *A*

Solution*:*

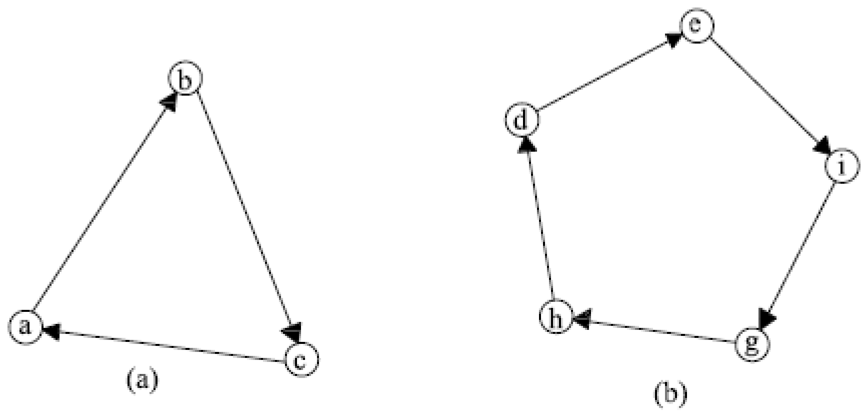
R = {*(a, a), (a, b), (b, c*), *(b, d), (c, d),(c, e), (d, b), (d, c*), *(e, a)*}.

The digraph of R is



Example 1.31:

Let *R* be a binary relation on A = *{a, b, c, d, e, f, g, h]* represented by the following two-component digraphs. Find the smallest integers *m* and *n* such that $m < n \text{ and } R^m = R^n$.



Solution:

From the figures (a) and (b), the relation R is

$$R = \{(a,b),(b,c),(c,a),(d,e),(e,f),(f,g),(g,h),(h,d)\}$$

The relation matrix for Figure (a) is

$$R = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$R^2 = R \ x \ R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R^3 = R^2 \; x \; R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^4 = R^3 \; x \; R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = R$$

For 3 nodes, $R^1 = R^4$

For 5 nodes, $R^1 = R^6$

Since the common multiple of 4 and 6 is 12, we conclude that

$$R^1 = R^{12}$$

Thus, m = 1 and n = 12.

Transitive closure:

Let R be a relation defined on a set A which is not transitive.

The *transitive* closure of the relation R is the smallest relation containing *R*, having the transitive property.

The *transitive closure* of *R*is just the *connectivity relation* $\boldsymbol{R^\infty}$ and it is also represented as transitive (R).

Results:

(i) Let *R* be a relation on a set *A*. Then R^∞ is the transitive closure of *R*.

(ii) Let R be a relation defined on a finite set A with $|A| = n$. Then,

$$R^\infty = R \cup R^2 \cup \ldots \cup R^n$$

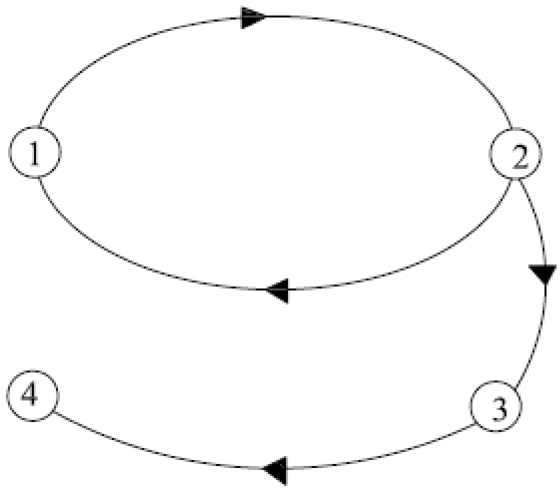
Example 1.32:

Find the transitive closure of the relation R = {(1,2),(2,3),(3,4),(2,1)} on the set A = {1,2,3,4}.

Solution:

(i) Graphical representation:

The digraph of *R* is shown in Figure 1.23.



Since R^{∞} is the transitive closure, we can proceed graphically by computing all the paths.

Paths exists from the vertex 1 to the vertices 1,2,3 and 4.

Thus the ordered pairs (1,1), (1,2),(1,3),(1,4) exists in R^{∞}.

Also, there exists paths from the vertex 2 to the vertices 1,2,3 and 4.

Thus the ordered pairs (2,1),(2,2),(2,3),(2,4) exists in R^{∞}.

The only other path formed is from vertex 3 to vertex 4.

Thus, Thus the ordered pair (3,4) exists in R^{∞}.

Hence, the transitive closure of R is

$$R^{\infty} = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,4)\}$$

(ii) Matrix Representation :

The relation matrix of R is

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, we compute the powers of M_R

$$(M_R)^2 = M_R \circ M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{x} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(M_R)^3 = (M_R)^2 \circ M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{x} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(M_R)^4 = (M_R)^3 \circ M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{x} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = {M_R}^2$$

Continuing in this way, it is observed that $(M_R)^n$ equal $(M_R)^2$, if n is even and equal $(M_R)^3$ if n is odd and greater than 1. Thus

$${M_R}^{\infty} = M_R \cup (M_R)^2 \cup (M_R)^3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R^{\infty} = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,4)\}$$

Example 1.33:

Let A=(1,2,3,4} and R={(1,2),(2,3) (2,4)} be a relation defined on A. Compute transitive closure of R.

Solution:

R = {(1,2),(2,3) (2,4)}

$$R^2 = R∘R = \{(1,2),(2,3),(2,4)\} ∘ \{(1,2),(2,3),(2,4)\} = \{(1,3),(1,4)\}R^3 = R^2∘R = \{(1,3),(1,4)\}∘\{(1,2),(2,3),(2,4)\} = \emptyset$$

$$R^4 = \emptyset$$

Transitive closure of R = $R^\infty = R \cup R^2 \cup R^3 \cup R^4$

= {(1,2), (2,3) (2,4)} ∪ $\{(1,3),(1,4)\} \cup \emptyset \cup \emptyset$

= {(1,2), (2,3) (2,4),(1,3) (1,4)}

Example 1.34:

Let the relation R be defined on the set A=(1,2,3} as R={(1,2), (2,3), (3,3)}.

Compute transitive(R).

Solution:

R = {(1,2), (2,3), (3,3)}.

$$\mathrm{R}^2 = \mathrm{R}∘\mathrm{R} = \{(1,2),(2,3),(3,3)\}∘\{(1,2),(2,3),(3,3)\}$$

$$= \{(1,3),(2,3),(3,3)\} \text{ and}$$

$$\mathrm{R}^3 = \mathrm{R}^2∘\mathrm{R} = \{(1,2),(2,3),(3,3)\}∘\{(1,2),(2,3),(3,3)\}$$

$$= \{(1,3),(2,3),(3,3)\}$$

transitive $(R) = R^\infty = R \cup R^2 \cup R^3$= {(1,2), (2,3), (3,3), (1,3)}

UNIT II

FUNCTIONS

2.1. INTRODUCTION TO FUNCTION

The concept of a function is extremely important in discrete mathematics. Functions are used in defining sequences and strings concretely.

Functions also express the time duration taken by a computer to solve problems of a given size.

Definition:

Let A and B be any two sets.

A **function** f from A to B is defined if for every element aϵA there exists a unique element bϵB, such that f(a)=bor(a,b) ϵf.

A function from A to B is denoted by f : A→B.

A is called the **domain** of f and B is called the **codomain** of f.

If f(a) = b. then b is the **image** of a and a is the **pre-image** of b.

The **range** of f is the set of all images of elements of A.

Functions are also called **mappings** or **transformations**.

A function f from A to B has following properties

i. Domain of f=A
ii. If (a,b) ϵ f and (a,c) ϵ f, then b=c.

Example 2.1:

Let A={1,2,3}, B=$\{a, b, c\}$ and f =$\{(1, a), (2, b), (3, c)\}$ is a function from A to B. Find the domain and range of the function f.

Solution:

Here, f(1)=a, f(2)=b, and f(3)=c

The domain of f is A and

The range of f is B.

Example 2.2:

Let A={1,2,3}, B=$\{a, b, c\}$and f=$\{(1, a), (2, b), (3, a)\}$. Is f a function from A to B? If yes, find the domain and range of f.

Solution:

Here, f(1) = a, f(2) = b, and f(3) = a.

∴f is a function from A to B

The domain of f is A and

The range of f is $\{a, b\}$

Example 2.3:

Let f: Z →N defined by f(x)=x^2+2. Find the domain, Codomain and the range of f.

Solution:

The domain of the function is Z.

The co-domain of f is N.

Now, f(0) = 2, f(1) = 3, f(2) = 6, and f(3) = 11, ….and

f(-1) = 3, f(-2) = 6, and f(-3) = 11, ….

The range of f is $\{2,3,6,11,18 \ldots \ldots\}$

Example 2.4:

Let A={1,2,3,4}, B=$\{a, b, c\}$ and f=$\{(1, a), (2, a), (3, b)\}$. Check whether f is a function or not?

Solution:

The domain of f = $\{1,2,3\}$ which is not equal to A.

∴ f is not a function.

Example 2.5:

Let R={$(1,a),(2,b),(3,c),(1,b)$}be a relation from A={1,2,3,4} to B={a,b,c}.Check whether f is a function or not?

Solution:

Here (1,a) and (1,b) are in R but a≠b.

∴ f is not a function.

Example 2.6:

Assume f as the function that assign the last two bits of a bit string of length 2 or greater to that string. i.e., f(11010)=10.

Solution:

The domain of f is the set of all bit strings of length 2 or greater.

The co-domain and range are the set is {00,01,10,11}.

2.2. Addition and Multiplication of function

Two real valued functions with the same domain can be added and multiplied.

Let f1 and f2 be two functions from AtoR.

Then f_1+f_2 and f_1f_2 are also functions from AtoR and are defined by

$(f_1 + f_2)x = f_1(x) + f_2(x)$

$(f_1 f_2)x = f_1(x) f_2(x)$

Example 2.7:

Given that f_1 and f_2 are functions from RtoR in which $f_1(x)=x$ and $f_2(x)=\left(\frac{1}{x}\right)-x$. Determine the function f_1+f_2 and f_1f_2.

Solution:

$$(f_1 + f_2)x = f_1(x) + f_2(x) = x + \frac{1}{x} - x = \frac{1}{x}$$

And $$(f_1 f_2)x = f_1(x).f_2(x) = x\left(\frac{1}{x} - x\right) = 1 - x^2$$

2.3. Classification of functions

1. One to one function (Injective function)

A function f:A→B is said to be one to one or injective, if distinct elements of domain set A have distinct images in co-domain set B.

f:A→ B is injective or one to one if

$a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$for all $a_1, a_2 \epsilon A$

In other words f: A →B is injective if

$f(a_1) = f(a_2) \Rightarrow \ \ a_1 = a_2 \forall\, a_1, a_2 \epsilon A$

Example 2.8:

Check whether the function f from $\{a, b, c, d\}$to $\{1,2,3,4,5\}$ with f(a) = 4, f(b) = 5, f(c) =1 and f(d)=3 is one to one or not?.

Solution:

The function f in one to one since f assigns different values at the four elements.

Example 2.9:

Examine the function $f(x) = x^2$from the set of integers to the set of integers for one to one

Soln:

The function $f(x) = x^2$ is not one to one sincef(1)=f(-1), but 1≠ -1.

Example 2.10:

Test the function $f(x) = x + 1$ for one to one.

Solution:

$$f(x) = f(y)$$
$$\Rightarrow x + 1 = y + 1$$
$$\Rightarrow x = y$$

(i.e) f(x)=f(y) ⇒ x=y

∴ f is a one to one function.

2. Onto (Surjective) function:

A function f from A to B is said to be an onto or surjectivefunction if for every element bϵB, there is an element aϵA such that$f(a) = b$.

Example 2.11 :

Let f be a function from $\{a, b, c, d\}$ to $\{1,2,3\}$ defined by f(a)=3, f(b)=2, f(c)=1 and f(d)=3. Check whether the function f is onto function or not.

Solution:

Here, all the three elements of the co-domain have pre-images in the domain.

So the function f is onto.

Example 2.12 :

Is the function f(x) = x+1 from the set of integers to the set of integers onto?

Solution:

Let f: Z→Z

f(x) = y

$\Rightarrow x + 1 = y$

$\Rightarrow x = y - 1$

(ie) for any yϵZ there exists an elements y-1ϵ Z such that f(y-1)=y.

∴f is onto.

3. One to one and onto : [Bijective function]

A function f : A → B is said to be a bijective function if f is both one to one and onto .

Example 2.13 :

Let f be a function form $\{a, b, c, d\}$to $\{1,2,3,4\}$ with f(a) = 4, f(b)=2, f(c)=1 and f(d)=3. Check whether the function is bijective or not.

Solution :

The domain of f =$\{a, b, c, d\}$.

The Codomain of f =$\{1,2,3,4\}$.

Function f is one to one because every elements of the domain have images.

Function f is onto since all the four elements of the co-domain have pre-images in the domain.

Function f is one to one and onto.

Hence f is a bijective function.

4. **Identity function**

A function f:A →Adefined by f(a)=a ∀ a∈A is called an identity function for A and it is denoted by I_A.

Also, I_A = {(a,a) : a∈A}.

The identity function I_A assigns each element to itself.

The function I_A is one – to – one and onto.

Hence, I_A is a bijective function.

5. **Constant function**

A function f : A → B is said to be a constant function if every element of A is assigned to the same element of B.

In other words, if the range of function f containsonly one element, then f is called a constant function.

Example 2.14 :

A function $f(x) = 5, \forall x \epsilon R$ is a constant function since

R_f = Range of f = {5}.

2.4. Composition of Function

Definition:

Let f: A →B and g:B→C be two functions.

The composition of functions f and g denoted by gof : A→C and it is defined as

(gof)(a) = g(f(a))

Note: The composition of function is not commutative .

i.e.,fog ≠ gof

Example 2.15 :

Let f be a function from the set {a,b,c} to the set {1,2,3} such that f(a)=3, f(b)=2, and f(c)=1.

Let g be a function from the set {a,b,c} to itself such that g(a)=b, g(b)=c, g(c)=a.

Determine the composition of f and g and also the composition of g of.

Solution:

The composition f and g is defined by

(fog)(a) = f(g(a)) =f(b)=2

(fog)(b)=f(g(b)) =f(c) =1

(fog)(c)=f(g(c))=f(a)=3

The composition of g and f is gofand it is not defined.

Example 2.16 :

Letf : Z →Zbe a function defined by f(x)=2x+3.

Let g : Z →Z be another function defined by g(x)=3x+2.

Determine the compositions f o g and g o f.

Solution:

The composition of f and g is fog : Z →Z and it is defined as

(fog)(x)=f(g(x))=f(3x+2)=2(3x+2)+ 3 =6x+7

Also the composition of g and f is g o f : Z →Z and it is defined as

$(\text{g o f})(\text{x}) = \text{g}(\text{f}(\text{x})) = \text{g}(2\text{x} + 3) = 3(2\text{x} + 3) + 2 = 6\text{x} + 11$

Example 2.17 :

Let f:R→Rbe defined by f(x) = x+1 and

g: R→R be defined as $g(x) = 2x^2 + 3$.

Find f o g and g of. Is fog=gof?

Solution :

f o g : Z →Z

$$(\text{fog})(\text{x}) = \text{f}\big(\text{g}(\text{x})\big) = \text{f}(2\text{x}^2 + 3) = 2\text{x}^2 + 3 + 1$$

$$= 2\text{x}^2 + 4$$

g o f : Z →Z

$$(\text{gof})(\text{x}) = \text{g}(\text{f}(\text{x})) = \text{g}(\text{x} + 1) = 2(\text{x} + 1)^2 + 3$$

$$= 2\,(\text{x}^2 + 2x + 1) + 3$$

$$= 2\,\text{x}^2 + 4\text{x} + 5$$

Here, f o g and g of are defined but $gof \neq fog$.

Example 2.18 :

Letf: A → B, g:B→C and h : C→D be three functions. Then show that ho(gof)=(hog)of.

Solution :

Given that f:A→B, g:B→C, and h : C →D

Then gof : A→C and hog : B→ D

Hence, ho(gof): A→D and (hog) of: A→D.

Let $x \epsilon A, y \epsilon B$ and $Z \epsilon C$ such that $f(x) = y$ and $g(y) = z$

Then, $[ho(gof)](x) = h[(gof)(x)] = h(g\big(f(x)\big) = h[g(y)] = h(z).$

$[(hog)]of(x) = (hog)[f(x)] = (hog)(y) = h[g(y)] = h(z).$

Thus

(hog)of=ho(gof), $\forall x \in A$

Theorem 2.1

Let f:A→ B and g: B→C be two functions.

I. If f and g are injective then gof is injective.

II. If f and g are surjective then gof is Surjective

Proof:

Given that f:A→ B and g:B→C are two functions.

∴ gof : A→ C

I. Given that f and g are injective

Let $x \epsilon A$ and $y \epsilon A$

Let $x \neq y$

$\Rightarrow f(x) \neq f(y)$ [∴ f is injective]

$\Rightarrow g\big(f(x)\big) \neq g(f(y))$ [∴ g is injective]

$\Rightarrow \big(gof(x)\big) \neq (gof)(y)$

∴ g o f is injective.

II. Let $z \in C$

To prove that gof is surjective find an element x∈ A such thatgof(x) = z.

Since f is surjective, for any y∈B there exists an element $x \epsilon A$ such that f(x)=y.

Since g is surjective, for anyz∈A there exists an element y∈B such that g(y)=z.

Now, $(gof)x = g\big(f(x)\big)$

$= g(y)$

= z

∴gof is surjective.

Corollary: 2.1

The converse part of the theorem is not true.

i.e., If gof is injective, it is not necessary that f and g are individually injective.

Theorem 2.2

The composition of any function with the identify function is the function itself.

Proof:

Let f: A→ B be any function.

Let $x \in A$ and $y \in B$ such that f(x)=y.

If I_A: A→A is the identify function, then

$I_A(x) = x \; \forall \, x \in A$

$(f \, o \, IA) : A \to B$

$(f o I A)(x) = f(I_A(x))$

$= f(x)$

$\therefore$ foIA= A

If I_B: B→Bis the identity function, than

$$I_B(y) = y \; \forall \, y \in B$$

$$I_B of : A \to B$$

$$(I_B of)(x) = I_B(f(x))$$

$$= f(x)$$

$$I_B of = f$$

Thus $f \, o I_A = I_B of = f$.

2.5 Inverse function

Let f:A→ B be a bijective function defined by f(a)=b, where a∈A and b∈B.

The inverse function of f is denoted by f^{-1} and f^{-1}: B→ A is defined by $f^{-1}(b) = a$ where a∈A and b∈B.

If we can define the inverse function of f , then f is said to be invertible.

If the function f is not a bijective function then we cannot define the inverse function of f.

Theorem 2.3

Let f:A→B be a bijective function and f is its inverse. For each x∈B

$$(fof^{-1})(x) = x$$

and for each $x \in A$ $(f^{-1}o\ f)(x) = x$

(i.e.,) fof^{-1}= I_B and $f^{-1}o\ f$ = I_A

Proof:

Given that f:A→ B is a bijective function.

∴ f^{-1}:B→ A

fof^{-1}: B→B

Let $x \in B$and f^{-1} (x)=z for any z ∈ A

f(z) = x

$(fof^{-1})(x) = f\left(f^{-1}(x)\right)$

= f(z)

= x

∴ fof^{-1}= I_B

Let $x \in A$ and $f(x) = z$ for any $z \in B$

∴ $f^{-1}(z) = x$

$f^{-1}$of : A→A

$f^{-1}of(x)$= $f^{-1}(f(x))$

$= f^{-1}(z)$

$= x$

∴ $f^{-1}of =$ I_A

Example: 2.19

Let A = {a.b,c}, B= {1,2,3} and f ={(a,1), (b,3), (c,2)}. Determine the inverse.

Solution :

f:A→B is both one-to-one and onto.

i,e., f is bijective.

f^{-1}: B→ A

f^{-1}={$(1, a), (2, c), (3, b)$}

Example: 2.20

Show that the function f(x)=x^3 and $g(x) = x^{1/3}$ for all x∈R are inverses of each other.

Solution:

$f(x) = x^3$

$g(x) = x^{1/3}$

$(fog)(x) = f\big(g(x)\big)$

$= f(x^{\frac{1}{3}})$

$= \left(x^{1/3}\right)^3$

= x

$\therefore fog = I$

$\therefore f = g^{-1}$

$(gof)(x) = g\big(f(x)\big)$

$= g(x^3)$

$= (x^3)^{1/3}$

= x

$\therefore g\ of = I$

$\therefore g\ = f^{-1}$

Thus the functions f and g are inverses of each other.

Example 2.21:

Let f: R→R be defined by f(x) = 3x- 4. Find a formula for f^{-1}.

Solution:

f: R→R

Given that f(x) = 3x-4

Consider f(x) = f(y)

⇒ 3x-4 =3y-4

⇒ 3x = 3y

⇒ x = y

i.e., f(x) = f(y) ⇒ x=y

∴ f is one- to- one.

Let y be the image of x under the function f.

f(x) = y

⇒3x-4 = y

⇒3x = y+4

⇒x = y + $\frac{4}{3}$

For any $y \in R$ there exists an element $y + 4/3 \in R$ such that

$$f\left(\frac{y+4}{3}\right) = y$$

$\therefore$f is onto.

f is one-to-one and onto.

$\therefore$f is bijective.

We have $f\left(\frac{y+4}{3}\right) = y$

$\therefore f^{-1}(y) = y + \frac{4}{3}$

$f^{-1}(x) = x + \frac{4}{3}$ which is the formula for f^{-1}.

Example2.22 :

Let X = {a, b, c}. Define $f : X \rightarrow X$ such that f = {(a,b) , (b,a) , (c,c) }.

Determine

i) f^{-1} ii) f^2 iii) f^3 iv) f^4

Solution:

i) f = { (a,b),(b,a),(c,c) }

f(a) = b	$f^{-1}(b) = a$
f(b) = a	$f^{-1}(a) = b$
f(c) = c	$f^{-1}(c) = c$

f^{-1} = {(b,a),(a,b),(c,c)}

ii) $f^2 = fof$

$f^2(a) = (fof)(a) = f(f(a)) = f(b) = a$

$f^2(b) = (fof)(b) = f(f(b)) = f(a) = b$

$f^2(c) = (fof)(c) = f(f(c)) = f(c) = c$

$\therefore f^2$ = {(a,a),(b,b),(c,c)}

iii) $f^3 = fof^2$

$f^3 = (fof^2)(a) = f(f^2(a)) = f(a) = b$

$f^3 = (fof^2)(b) = f(f^2(b)) = f(b) = a$

$f^3 = (fof^2)(c) = f(f^2(c)) = f(c) = c$

$\therefore f^3$ = {(a,b),(b,a),(c,c)}

iv) $f^4 = fof^3$

$f^4(a)=(fof^3)(a)=f(f^3(a))=f(b)=a$

$f^4(b) = (fof^3)(b) = f(f^3(b)) = f(b) = a$

$f^4(c) = (fof^3)(c) = f(f^3(c)) = f(c) = c$

f^4={(a,a),(b,b),(c,c)}

Example 2.23:

Let f be the function from the set of integers to the set of integers such that f(x)=x+1. Is f is invertible and if so, find its inverse.

Solution:

$f : Z \rightarrow Z$

$f(x)=x+1$

Consider that $f(x)=f(y)$

$\Longrightarrow x+1=y+1$

$\Longrightarrow x = y$

i,e., $f(x)=f(y) \Longrightarrow x=y$

∴f is one-to-one.

Let y be the image of xunder the function f.

$f(x)=y$

$x+1=y$

$x=y-1$

For any $y \in Z$ there exits an element $y-1 \in Z$ such that $f(y-1)=y$

∴ f is onto.

f is both one-to-one and onto.

∴f is bijective.

∴f is invertible.

We have $f(y-1)=y$

$\therefore f^{-1}(y)=y-1$

$f^{-1}(x)=x-1$

Inverse of f is f^{-1} and $f^{-1}: Z \rightarrow Z$ defined by $f^{-1}(x)=x-1$.

Theorem 2.4:

If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are bijections then $(gof)^{-1}=f^{-1}og^{-1}$

Proof:

Since f and g are bijections, gof is a bijection.

$gof: X \rightarrow Z$

$(gof)^{-1}: Z \rightarrow X$ is also a bijection.

$f: X \rightarrow Y$ is a bijection.

$\therefore f^{-1}: Y \rightarrow X$ is a bijection.

$g: Y \rightarrow Z$ is a bijection.

$\therefore g^{-1}: Z \rightarrow Y$ is a bijection.

$\therefore f^{-1} o g^{-1}: Z \rightarrow X$ is a bijection.

Now let $z \in Z$

Since g is onto, there exists an element $y \in Y$ such $g(y)=z$.

$g^{-1}(z)=y$

Since f is onto, for any $y \in Y$, there exists an element $x \in X$ such that $f(x)=y$

$\therefore f^{-1}(y)=x$

$(f^{-1} o g^{-1})(z)=f^{-1}(g^{-1}(z))=f^{-1}(y)=x$

i,e$(f^{-1} o g^{-1})(z)=x$ ……………. (1)

Now $(gof)(x)$

$=g(f(x))$

$=g(y)$

$=z$

$\therefore (gof)^{-1}(z)=x$ ………………(2)

From (1) & (2) we get

$(gof)^{-1}=f^{-1} o g^{-1}$

UNIT III

MATHEMATICAL LOGIC

3.1 Introduction:

Mathematical logic emerged in the mid-19th century as a subfield of mathematics independent of the traditional study of logic**.** **Mathematical logic** is a subfield of **mathematics** exploring the applications of formal **logic** to **mathematics**. **Logic** is the basis of all mathematical and automated reasoning. The **logical reasoning**, also known as critical thinking or analytic **reasoning**, involves one's ability to isolate and identify the various components of any given argument.

3.2. STATEMENT (Propositions)

A Statement (or a proposition) is a declarative sentence (i.e, a sentence that declares afact) which is either True or False but not both and which is also sufficiently objective, meaningful and precise.

The truth or falsity of a statement is called its truth value.

The truth values "True and False" of a statement are denoted by True and False respectively.

The value of a statement if true is denoted by 1 and false if expressed by 0.

For example:

Consider the following sentences.

(i) Tamil Nadu is in India.
(ii) 7+2=9.
(iii) 5< 10
(iv) Bangalore is in West Bengal
(v) X+2= 7
(vi) Where are you going?
(vii) Roses are red.
(viii) Go to bed.

The sentences (i),(ii),(iii),(iv) and (vii) are statements.

Among these (iv) is false and others are true.

(v)is not a proposition (or a statement),since it is neither true nor false.

(vi) is a question, it is not a statement.

(viii) is not a statement but it is a command only.

Laws of Formal Logic

The two famous laws of formal logic are

1. Law of contradiction :For every proposition pit is not the same notion that p is both true and false.
2. Law of intermediate exclusion
   If p is a statement (proposition), then either p is true or false is no possibility of intermediate exclusion.

3.3. Basic Set of Logical operators/operations:

The three basic logical operators/operations are conjuction (Λ),disjunction (V), and negation (~) which corresponds to the English words like 'and','or', and 'not',respectively.

1. **Conjuction:(AND / Λ)**

If p and q are any two positions, then the conjunction of p and q is denoted by pΛq .

The truth value of pΛq is true if p is true and q is true.Otherwise pΛq is also false.

The Truth Table for pΛq is given in Table 3.1

P	Q	pΛq
T	T	T
T	F	F
F	T	F
F	F	F

Example 3.1:

Find the conjunction of the proposition p and q when p is the proposition 'Today is Saturday' and q is the proposition ' It is raining heavily today'.

Solution:

p: 'Today is Saturday' .

q: ' It is raining heavily today'

pΛq :'Today is Saturday and it is raining heavily today".

2. Disjunction(OR,pvq)

If p and q are any two positions, then the disjunction of p and q is denoted by pVq and it is read as 'p or q'.

The truth value of pVq is true if any one of the propositions p or q is true. If p and q are false, then pVq is false.

The Truth Table for pVq is given in Table 3.2

p	q	pVq
T	T	T
T	F	T
F	T	T
F	F	F

Example3.2 :

Find the disjunction of the propositions p and q where p is the proposition "Today is Saturday" and q is the proposition 'It is raining heavily today'.

Solution:

p: 'Today is Saturday' .

q: ’ It is raining heavily today’

pVq: ‘Today is Saturday or it is raining heavily today’.

Example 3.3 :

Let p be ‘Suja Speaks Tamil’ and let q be ‘Suja Speaks English’. Give a simple verbal sentence which describes each of the following.

(i) pVq (ii) p^q

Solution:

p : ‘Suja Speaks Tamil

q :Suja Speaks English’’

(i) pVq:Suja Speaks Tamil or English.

(ii) p^q:Suja Speaks Tamil and English.

Example3.4 :

Assign a truth value to each of the following statements.

(i) $5+5=10 \vee 1>2$

(ii) $6\times4=21 \vee 2+7=10$.

Solution:

(i) True, since one of its components is true. ,i.e.,5+5=10 is true.

(ii) False, since both the components are false.

3. Negation(NOT,~p)

The negation of proposition p is denoted by ~p and it is read as ‘not p’.

The negation of a proposition can be formed by stating ‘It is not the case that’ or ‘It is false that ‘.

The truth value of ~p is represented in the following table.

p	~p
T	F
F	T

Example3.5 :

Find the negation of the following statements

(i) Kolkata is in India.
(ii) 4+4=9.

Solution :

(i) It is not the case that Kolkata is in India.

Or Kolkata is not in India

Or It is not the case that Kolkata is in India.

(ii) $4+4 \neq 9$.

Example 3.6 :

Find the negation of the following propositions.

(i) Today is Sunday.
(ii) It is a rainy day.
(iii) If it snows, Mani does not drive the car.

Solution:

(i) Today is not Sunday.
(ii) It is not a rainy day.
(iii) It snows and Mani drives the car.

Example 3.7 :

Let p: Jemila is tall and q:Jemila is beautiful..Write the following statements in symbolic form.

(i) Jemila is tall and beautiful.
(ii) Jemila is tall but not beautiful
(iii) It is false that Jemila is short or beautiful
(iv) Jemila is tall or Jemila is short and beautiful

Solution:

(i) $p \wedge q$
(ii) $p \wedge \sim q$
(iii) $\sim(\sim p \vee q)$
(iv) $p \vee (\sim p \wedge q)$

3.4. Proposition and truth tables:

Let P(p,q) be a proposition constructed from logical variables p,q….. which take on the value TRUE(T) or (FALSE)(F), and which operate on the logical connectives $\wedge, \vee, \sim$. *S*uch an expression is called a proposition.

Example3.8 :

Construct the truth table for $\sim(p \vee q)$.

Solution :

p	q	$p \vee q$	$\sim(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Connectives:

The statements which do not contain any of the connectives are called **atomic statements** or **simple statements** or **Primitive Statements**.

The common logical connectives used are **negation(~), and(∧), or(∨), if…..then(→or⇒), if and only if(↔ or ⇔)** and **equivalence(≡).**

Example 3.8 :

Write the following statements in Symbolic form.

(i) If Anand is not in a good mood or he is not busy, then he will go to Kharagpur.

(ii) If Santhosh knows Object-Oriented Programming and Oracle, then he will get a job.

Solution:

(i) Let p: Anand is in good mood,
q:Anand is busy and
r:Anand will go to kharagpur

The statement in symbolic form is $(\sim p \vee \sim q) \rightarrow r$

(ii) Let p: Santhosh knows Object-Oriented Programming
q:Santhosh knows Oracle

r:Santhosh will get a job

The statement in symbolic form is $(p \wedge q) \rightarrow r$

Example3.9 :

Let p: Babin is rich, q:Babin is happy. Write simple verbal sentences which describes each of the following statements.

(i)$p \vee q$ (ii)$p \wedge q$ (iii)$q \rightarrow p$ (iv)$p \vee \sim q$

(v)$q \leftrightarrow p$ (vi) $\sim p \rightarrow q$ (vii) $\sim\sim p$ (viii)$(\sim p \wedge q) \rightarrow p$

Solution:

(i) Babin is rich or Babin is happy.
(ii) Babin is rich and Babin is happy.
(iii) If Babin is happy then Babin is rich.
(iv) Babin is rich or Babin is not happy.
(v) Babin is happy if and only if Babin is rich.
(vi) If Babin is not rich then Babin is happy.
(vii) It is not true that Babin is not rich.
(viii) If Babin is not rich and happy then Babin is rich.

Compound Propositions:

Compound or **composite** propositions (statements) are composed from sub-propositions by means of logical operators or connectives.

Example3.10 :

(i) Write a compound proposition with sub-propositions 'Sojan is intelligent' and Sojan studies every night'.
(ii) Write a compound proposition with sub-propositions' The sun is shining' and 'the sky is blue'.

Solution :

(i) Sojan is intelligent or studies every night.
(ii) 'The sun is shining and the sky is blue'

Example 3.11 :

Construct truth tables for each of the following compound propositions.

(i) $(p\wedge q)\vee(p\wedge r)$ (ii) $\sim(p\vee q)\vee(\sim p\wedge\sim q)$

Solution :

(i) $(p\wedge q)\vee(p\wedge r)$

p	q	R	p∧q	p∧r	(p∧q)∨(p∧r)
F	F	F	F	F	F
F	F	T	F	F	F
F	T	F	F	F	F
F	T	T	F	F	F
T	F	F	F	F	F
T	F	T	F	T	T
T	T	F	T	F	T
T	T	T	T	T	T

(ii)) ~(p∨q)∨(~p∧~q)

p	q	~p	~q	p∨q	~(p∨q)	(~p∧~q)	~(p∨q)∨(~p∧~q)
F	F	T	T	F	T	T	T
F	T	T	F	T	F	F	F
T	F	F	T	T	F	F	F
T	T	F	F	T	F	F	F

Example 3.12 :

Construct the truth table of p∧ (q∨r)

Solution :

P	q	r	q∨r	p∧(q∨r)
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

Conditional Statement:

If p and q are any two statements, then the statement p→q which is read as 'if p then q' is called a conditional statement.

If p is true and q is false, then the conditional statement p→q is false.

Otherwise p→q is true.

The statement p is called the **antecedent** and the statement q is called the **consequent** (or **conclusion**).

p→q is interpreted as 'p is sufficient for q' or 'q whenever p'.

The truth table for p→q is given in table

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Converse, Contrapositive and Inverse.

The **converse** of $p \rightarrow q$ is the proposition $q \rightarrow p$.

The **contrapositive** of $p \rightarrow q$ is the proposition $\sim q \rightarrow \sim p$.

The **inverse** of $p \rightarrow q$ is the proposition $\sim p \rightarrow \sim q$.

Example3.13 :

Write the conditional statement for the following statements.

(i) Let p:Anub is a graduate and
q :Anub is a lawyer

(ii) Let p:The function is differentiable and
q:The function is continuous.

Solution :

(i) $p \rightarrow q$:If Anub is a graduate, then she is a lawyer.

(ii) $p \rightarrow q$: If the function is differentiable, then it is continuous.

Example 3.14 :

Determine the contrapositive, the converse and the inverse of the conditional statement 'The Team A wins whenever it is raining'.

Solution:

Let p:It is raining and
q : The Team A wins.

The given statement can be modified as

'If it is raining, then the Team A wins'.

It is the conditional statement $p \rightarrow q$.

The contra positive of this conditional statement is

$\sim q \rightarrow \sim p$: If the Team A does not win, then it is not raining.

The converse is

$q \rightarrow p$:If the Team A wins, then it is raining.

The inverse is

$\sim p \rightarrow \sim q$: If it is not raining, then the Team A does not win.

Biconditional Statement:

A statement of the form 'p if and only if q' is called a biconditional statement. It is denoted by $p \leftrightarrow q$.

If p and q have the same truth value, then $p \leftrightarrow q$ is true.

If p and q have distinct truth values, then $p \leftrightarrow q$ is false.

The truth table for $p \leftrightarrow q$ is shown in table

P	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example3.15 :

Write any two biconditional statements.

Solution :

(i) An integer is even if and only if it is divisible by 2.
(ii) Two lines are parallel if and only if they have the same slope.

Example3.16 :

Show that $p \Rightarrow q$ is the same as $\sim q \Rightarrow \sim p$.

i.e., The contrapositive $\sim q \rightarrow \sim p$ of a conditional statement $p \rightarrow q$ always has the same truth value as $p \rightarrow q$.

Solution:

p	q	$\sim p$	$\sim q$	$\sim q \Rightarrow \sim p$	$p \Rightarrow q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

3.5 Algebra of propositions :

Various laws useful to simplify the propositions are listed in the following table.

1. (a)$p \lor p \equiv p$	Idempotent laws	(b)$p \land p \equiv p$
2. (a)$(p \lor q) \lor r \equiv p \lor (q \lor r)$	Associative laws	(b)$(p \land q) \land r \equiv p \land (q \land r)$
3. (a)$p \lor q \equiv q \lor p$	Commutative laws	(b)$p \land q \equiv q \land p$
4. (a)$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$	Distributive laws	(b)$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
5. (a)$p \lor T \equiv T$	Identity laws	(b)$p \land T \equiv p$
6. (a)$p \lor F \equiv p$	Identity laws	(b)$p \land F \equiv F$
7. (a)$p \lor \sim p \equiv T$	Complement laws	(b)$p \land \sim p \equiv F$
8. (a) $\sim T \equiv F$		(b) $\sim F \equiv T$
9. $\sim\sim p = p$	Involution law	
10. (a) $\sim(p \lor q) \equiv \sim p \land \sim q$	Demorgan's law	(b) $\sim(p \land q) \equiv \sim p \lor \sim q$

3.6. Tautologies and Contradictions:

A propositions P(p,q) is called a **tautology** if the last column of their truth tables contain only T. i.e., If the propositions are true for any truth values of their variables, then such propositions are called **tautologies.**

A proposition P(p,q) is called a **contradiction** if it contains only F in the last column of its truth table.

A proposition that is neither a tautology nor a contradiction is called a **contingency**.

Example3.17 :

Show that the proposition $p \vee \sim p$ is a tautology.

Solution :

P	~p	$p \vee \sim p$
T	F	T
F	T	T

Since the truth value of $p \vee \sim p$ is true for all values of p, the proposition is a tautology.

Example3.18 :

Show that the proposition $p \wedge \sim p$ is a tautology.

Solution :

P	~p	$p \wedge \sim p$
T	F	F
F	T	F

Since the truth value of $p \wedge \sim p$ is true for all values of p, the proposition is a tautology.

Example3.19 :

Verify that the proposition $p \vee \sim(p \wedge q)$ is a tautology.

Solution :

p	q	p∧q	~(p∧q)	p∨~(p∧q)
F	F	F	T	T
F	T	F	T	T
T	F	F	T	T
T	T	T	F	T

Since the truth value of p∨~(p∧q) is true for all values of p, the proposition is a tautology.

Example3.20 :

Show that the following $(p \wedge \sim q) \vee \sim(p \wedge \sim q)$ is a tautology.

Solution :

p	q	~q	$p \wedge \sim q$	$\sim(p \wedge \sim q)$	$(p \wedge \sim q) \vee \sim(p \wedge \sim q)$
T	T	F	T	F	T
T	F	T	F	T	T
F	T	F	T	F	T

F	F	T	T	F	T

Since the truth value of $(p\wedge\sim q)\vee\sim(p\wedge\sim q)$ is true for all values of p, the proposition is a tautology.

Example 3.21 :

Show that $p \to (q \to r) \Leftrightarrow p \to (\sim q\vee r) \Leftrightarrow (\sim p\wedge q)\vee r$

Solution :

$$p \to (q \to r)$$

$\Leftrightarrow p \to (\sim q\vee r)$ $[\,Since\; p \to q \equiv \sim p\vee q]$

$\Leftrightarrow \sim p\vee(\sim q\vee r)[\,Since\;\; p \to q \equiv \sim p\vee q]$

$\Leftrightarrow (\sim p \vee\sim q)\vee r[Since\; by\; Associative\; law\; p\vee(q\vee r) \equiv (p\vee q)\vee r]$

$\Leftrightarrow \sim(p\wedge q) \vee r[By\; De - Morgan's\; law \sim(p\wedge q) \equiv \sim p\vee\sim q]$

$\Leftrightarrow (p\wedge q) \to r$ $[Since\; p \to q \equiv \sim p\vee q]$

Hence

$$p \to (q \to r) \Leftrightarrow p \to (\sim q\vee r) \Leftrightarrow (\sim p\wedge q)\vee r$$

Example 3.22:

Show that $(p \to q)\wedge(r \to q) \Leftrightarrow (p\vee r) \to q$.

Solution:

$p \to q \Leftrightarrow \sim p\vee q$

$r \to q \Leftrightarrow \sim r\vee q$

Consider $(p \to q) \wedge (r \to q)$

$\Leftrightarrow (\sim p \vee q) \wedge (\sim r \vee q)$

$\Leftrightarrow (\sim p \wedge \sim r) \vee q$ [$Since\ (p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$]

$\Leftrightarrow \sim (p \vee r) \vee q$ [$Since\ \sim (p \vee q) \equiv \sim p \wedge \sim q$]

$\Leftrightarrow (p \vee r) \longrightarrow q$ [$Since\ p \to q \equiv \sim p \vee q$

Hence $(p \to q) \wedge (r \to q) \Leftrightarrow (p \vee r) \to q.$

3.7. Logical Equivalence:

Two propositions P(p,q……….) and Q(p,q…………….) are said to be **logically equivalent** or simply **equivalent** or **equal** denoted by P(p,q…..) ≡Q(p,q…..) if they have the identical truth tables.

The propositions P(p,q) and Q(p,q…..) are **logically equivalent** if P↔Q is a tautology.

The equivalence of P and Q is also denoted by P⟺Q.

State and prove De-Morgan's law:

If p and q are any two propositions, then

I. $\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$
II. $\sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$

Proof :

I. $\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$

P	q	~p	~q	Pvq	~(p∨q)	(~p)∧(~q)
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Since the two columns headed by $\sim(p \vee q)$and $(\sim p) \wedge (\sim q)$ of the truth table are identical,

$$\sim(p \vee q) \equiv (\sim p) \wedge (\sim q).$$

II $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

P	q	~p	~q	p ∧ q	$\sim(p \wedge q)$	$(\sim p) \vee (\sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Since the two columns headed by $\sim(p \wedge q)$and $(\sim p) \vee (\sim q)$of the truth table are identical,

$$\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$$

Example3.23 :

Show that p∧(q∨r) is equivalent to (p∧q)∨(p∧r)

Solution :

p	q	r	q∨r	p∧(q∨r)	p∧q	p∧r	(p∧q)∨(p∧r)
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Since the two columns headed by p∧(q∨r) and (p∧q)v(p∧r) of the truth table are identical,

p∧(q∨r) ≡ (p∧q)∨(p∧r)

Example 3.24 :

Show that $p \Leftrightarrow q$ and $(p \Rightarrow q) \wedge (q \Rightarrow p)$ are equivalent.

p	Q	$p \Leftrightarrow q$	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T	T

T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Since the columns headed by $p \Leftrightarrow q$ and $(p \Longrightarrow q) \wedge (q \Longrightarrow p)$ of the truth table are identical,

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

Example 3.25 :

Among the two restaurants next to each other,one has a sign that says'Good food is not cheap' and the other has a sign that 'cheap food is not good'.

Investigate the signs regarding their equivalence.

Solution:

Let p :Food is good and

q:Food is cheap.

The first sign says $p \rightarrow \sim q$ and the second one says $q \rightarrow \sim p$.

p	q	~p	~q	$p \rightarrow \sim q$	$q \rightarrow \sim p$
F	F	T	T	T	T
F	T	T	F	T	T
T	F	F	T	T	T

T	T	F	F	F	F

From the truth table we conclude that, both the signs are equivalent.

3.8. Normal Forms:

Some of the basic normal forms are:

I. Disjunctive normal form(dnf)
II. Conjunctive normal form(cnf)

I. Disjunctive normal form(dnf)

In a logical expression, a product of the variables and their negations is called an **elementary product** or **minterm**.

Example :$P\wedge\sim R$,$Q\wedge P\wedge\sim R$ etc.

An elementary product is identically false if and only if it contains atleast one pair of factors in which one is the negation of the other.

A logical expression is called a **disjunctive normal form**,abbreviated as **dnf**,if it is a **sum of elementary products**.

II. Conjunctive Normal form (cnf)

In a logical expression the sum of the variables and their negation is called an **elementary sum**.

Example :$P\vee\sim Q$,$\sim P\vee\sim Q\vee\sim R$

An elementary sum is identically true if and only if it contains atleast one pair of factors in which one is the negation of the other.

A logical expression is called a **conjunctive normal form**, abbreviated as **cnf**, if it is a **product of elementary sums**.

9.Find the inverse of the following

$$A=\begin{pmatrix}1 & -2 & 2\\ 2 & -3 & 6\\ 1 & 1 & 7\end{pmatrix}$$

$$|A| = 1(-21-6) + 2(14-6) + 2(2+3)$$

=1(-27)+2(8)+2(5)

=-27+16+10

=-1

$\therefore A\ is\ a\ knon\ singular\ matrix$

$\therefore\ A^{-1}\ exists$

Cofactors of elements of |A| are

A_{11}=(-21-6)=-27

A_{12}=-(14-6)=-8

A_{13}=(2+3)=5

A_{21}=-(-14-2)=16

A_{22}=(7-2)=5

A_{23}=-(1+2)=-3

A_{31}=(-12+6)=-6

A_{32}=-(6-4)=-2

A_{33}=(-3+4)=1

Adj A=$(Aij)^T$

$$\begin{pmatrix}-27 & 16 & -6\\ -8 & 5 & -2\\ 5 & -3 & 1\end{pmatrix} = adj\ A$$

$$A^{-1} = \frac{1}{|A|}.adj\ A$$

$=\frac{1}{2}\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}$

$A^{-1}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\0 & 1\end{pmatrix}$

Find the inverse of the matrix $\begin{pmatrix}1 & 1 & 3\\1 & 3 & -3\\-2 & -4 & -4\end{pmatrix}$

Solution:

Let $A=\begin{pmatrix}1 & 1 & 3\\1 & 3 & -3\\-2 & -4 & -4\end{pmatrix}$

$|A|=1(-12-12)-1(-4-6)+3(-4+6)$

$=1(-24)-1(-10)+3(2)$

$=24-1(-10)+6$

$=-24+10+6$

$=-8$

$|A|\neq 0$

$\therefore$ A is a non singular matrix

A^{-1} exists

Cofactors of elements of |A| are

$A_{11}=\begin{vmatrix}3 & -3\\-4 & -4\end{vmatrix}=-12-12=-24$

$A_{12}=-\begin{vmatrix}1 & -3\\-2 & 4\end{vmatrix}=-(-4+6)=-(-10)=10$

$A_{13}=\begin{vmatrix}1 & 3\\-2 & -4\end{vmatrix}=-4+6=2$

$A_{21}=-\begin{vmatrix}1 & 3\\-4 & -4\end{vmatrix}=-(-4+12)=-18$

$A_{22}=\begin{vmatrix}1 & 3\\-2 & -4\end{vmatrix}=-4+6=2$

$A_{23}=-\begin{vmatrix}1 & 1\\ -2 & -4\end{vmatrix}=$

A_{31}=(-12+6)=-6

A_{32}=-(6-4)=-2

A_{33}=(-3+4)=1

adj A=$(Aij)^T$

$$\begin{pmatrix}-27 & 16 & -6\\ -8 & 5 & -2\\ 5 & -3 & 1\end{pmatrix}=\text{adj } A$$

$$A^{-1}=\frac{1}{|A|}.adj\ A$$

$$=\frac{1}{-1}\begin{pmatrix}-27 & 16 & -6\\ -8 & 5 & -2\\ 5 & -3 & 1\end{pmatrix}$$

$$=\begin{pmatrix}27 & -16 & 6\\ 8 & -5 & 2\\ -5 & 3 & 1\end{pmatrix}$$

(ii)$B=\begin{pmatrix}1 & 3 & -4\\ 1 & 5 & -1\\ 3 & 13 & -6\end{pmatrix}$

|B|=1(-30+13)-3(-6+3)-4(13-15)

=1(-17)-3(3)-4(-2)

=-17+9+8

|B|=0.

B is a singular matrix

∴ B^{-1} does not exists.

Example: 5.14

If $A=\begin{pmatrix}2 & 3\\ 4 & 8\end{pmatrix}$ verify that A(adj A)=(adj A)A=det(A)I

Solution:

Let A=$\begin{pmatrix} 2 & 3 \\ 4 & 8 \end{pmatrix}$

|A|=16-12

|A|=4

adj A=$\begin{pmatrix} 8 & -3 \\ -4 & 2 \end{pmatrix}$

A×(adj A)=$\begin{pmatrix} 2 & 3 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} 8 & -3 \\ -4 & 2 \end{pmatrix}$

$\begin{pmatrix} 16-12 & -6+6 \\ 32-32 & -12+16 \end{pmatrix}$

=$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

(adj A)×A=$\begin{pmatrix} 8 & -3 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 8 \end{pmatrix}$

$= \begin{pmatrix} 16-12 & 24-24 \\ -8+8 & -12+16 \end{pmatrix}$

$= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

$4I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

∴A(adj A)=adj A(A)=det I

Hence it proved

Determinant of a matrix:

Determinant of a square matrix of may be denoted by det A or |A| or Δ.Determinant of a square matrix of order 1,ie.A=[a]

Det A=|A|=a

Determinant of 2×2 matrix(i.e.,of order 2)

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$|A|=a_{11}a_{22}-a_{21}-a_{12}$

Minor and co-factor:

Let $A=[aij]_{m\times n}$.The minor of an element aij of determinant of a matrix A is the determinant formed by suppreving the i^{th} row and the j^{th} coloumn in which the element aij exists.

The minor of the element aij is denoted by Mij.

The minor of the element of a determinant of n is a determinant of order(n-1).

The cofactor of an element aij is denoted by Aij is defined as $Aij=(-1)^{i+j}Mij$.

For example: The minor and co-factor of the elements a_{11},a_{22} and a_{33} of the element.

$$\Delta=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Can be obtained as follows

M_{11}(minor of all)$=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$=a_{22}a_{33}-a_{23}a_{32}$

A_{11}(co-factor of a_{11})$=(-1)^{2+2}\ M_{22}$

M_{22}(Monior of a_{32})$=\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

$=a_{11}a_{23}-a_{13}a_{31}$

A_{22}(cofactor of a_{22})$=(-1)^{2+2}\ M_{22}$

$=+(a_{11}a_{23}-a_{13}a_{31})$

M_{32}(monior of a_{32})= $\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

$=a_{11}a_{23}-a_{13}a_{21}$

A_{32}(co-factor of a_{32})=$(-1)^{3+2}$ M_{32}=$-(a_{11}a_{23}-a_{13}a_{21}$_

Expansion of the Determinant:

The determinant(Δ)of a matrix A can be expressed as the sum of the products of elements of any row(or coloumn) by their corresponding co-factors.

$\Delta=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\Delta= \sum_{j=1}^{3} (-1)^{i+j} \ aijMij$$

$$= \sum_{j=1}^{3} aij\{(-1)^{i+j} \ Mij\}$$

UNIT IV

MATRIX ALGEBRA

4.1. INTRODUCTION

Matrix algebra plays an important and powerful role in quantitative analysis of management decisions in several disciplines such as production, marketing, finance, economics, computer science, discrete mathematics, network analysis, Markov models, input-output models and some statistical models. All these models are built by establishing a system of linear equations.

Matrices are useful because they enable us to consider an array of numbers as single object, denote it by a single symbol, and perform operations with these symbols in a precise form.

Definition of a Matrix

A rectangular array of entries arranged in m rows and n columns is called a **matrix of order m by n**, written as **m×n matrix**.

A matrix is usually denoted by a boldface capital letter enclosed within brackets for example A or $[a_{ij}]$ respectively.

a_{ij} represents the element in the i^{th} row and the j^{th} column of a matrix A.

$A=[a_{ij}]_{m\times n}, 1\leq i \leq m\ ,\ 1\leq j\leq n.$

A is a matrix of order m×n.

In general an m×n matrix A may be written as

$$A = \begin{pmatrix} a11 & a12 & \dots a1n \\ a21 & a22 & \dots a2n \\ \dots & \dots & \dots\ \dots \\ am1 & am2 & \dots amn \end{pmatrix}$$

The i^{th} row consists of the entries

$a_{i1}, a_{i2}, \ldots\ldots\ldots\ldots\ldots\ldots a_{in}$

The j^{th} column consists of the entries

$a_{1j}, a_{2j}, \ldots\ldots\ldots\ldots\ldots\ldots a_{mj}$

Example : $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 5 & 7 \end{pmatrix}$ is a matrix of order 2 x 3.

NOTE : A matrix of order m×n contains mn elements.

4.2. Types of Matrices

Let $A=[a_{ij}]_{m\times n}$ …………… (i)

1. Rectangular and Square Matrices

If m≠ n ,then the matrix A is a **rectangular matrix** of order m×n.

If m=n, then the matrix A is a Square matrix of order n.

If $A=[a_{ij}]_{n\times n}$ is a Square matrix, then the principal/leading diagonal elements are a_{ii}.

The diagonal elements of Aare $a_{11},a_{22},a_{33},\ldots\ldots\ldots,a_{nn}$.

The sum of principal diagonal elements of a square matrix is known as ***trace of the matrix***.

Trace of $A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots + a_{nn}$

Example:1

$\begin{pmatrix} 2 & 2 & 3 \\ 3 & 5 & 7 \end{pmatrix}$ is a rectangular matrix of order 2×3.

Example:2

A=[5], $B = \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$,and C = $\begin{pmatrix} 3 & 1 & 5 \\ 1 & 5 & 1 \\ 7 & 5 & 2 \end{pmatrix}$

A is a square matrix of order 1.

B is a square matrix of order 2and

C is a square matrix of order 3.

Trace of B=2+4=6

Trace of C=3+5+2=10

2. Row matrix or a Row Vector

A matrix having only one row and any finite number of columns is called a row matrix or a row vector.

If m=1 then the matrix A is a 1×n matrix.

Example:

(1 5 3 4) is a row matrix of order 1×4 and

(5) is a row matrix of order 1×1.

3. Column Matrix or a Column Vector

A matrix having only one column and any finite number of rows is called a column matrix or a column vector.

If n=1 in equation (1) then the matrix A is an m×1 matrix.

Example :

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a coloumn matrix of order 3×1.

4. Zero or Null matrix

A matrix whose elements are all zero is called a Zero matrix or Null matrix. A zero matrix of order m×n is denoted by $O_{m\times n}$.

Example:

(0 0) is a zero matrix of order 1×2 and is written as $0_{1\times 2}$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a null matrix of order 2×2 and is represented as $0_{2\times 2}$

5. Diagonal Matrix

A square matrix of order n having non-zero elements on the main diagonal is called a diagonal matrix of order n.

Example 1 :

$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ is a diagonal matrix of order 2×2 .

Example2 :

$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ is a diagonal matrix of order 3×3.

6. **Scalar Matrix**

A Square matrix in which every non-diagonal element is zero and all diagonal elements are equal is called a scalar matrix.

Example 1 : $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$is a Scalar matrix of order 2×2

Example2 : $\begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$ is a Scalar matrix of order 3.

7. **Unit matrix or Identity matrix**

If A is a square matrix of order n in which every non-diagonal element is zero and every diagonal element is 1, then the matrix A is called a *unit matrix* or *identity matrix* of order n and it is denoted by I_n.

Example:

I_1=[1] is the identity matrix of order 1.

$I_2= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$is the identity matrix of order 2.

$I_3= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the identity matrix of order 3.

8. **Comparable Matrices**

Two matrices $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are said to be comparable matrices if they are of the same order.

Example:

The matrices $\begin{pmatrix} 4 & 1 & -2 \\ 3 & 2 & -4 \end{pmatrix}$ and $\begin{pmatrix} 6 & -3 & -6 \\ -1 & 0 & 2 \end{pmatrix}$ are comparable because both are of the order 2×3.

9. Equal matrices

Two matrices $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{p\times q}$ are said to be equal, written as A=B, if they are of the same order and their corresponding elements are equal.

Example:

Let $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B=\begin{pmatrix} 4 & 5 \\ 1 & -1 \end{pmatrix}$

A=B if and only if

$a_{11}=4$, $a_{12}=5$

$a_{21}=1$, $a_{22}= -1$

10. Upper triangular Matrix

If all elements below the main diagonal are zero, then the matrix A is called an *upper triangular* matrix.

Example: $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ is an upper triangular matrix of order 3.

11. Lower triangular matrix

If all elements above the main diagonal are zero, then the matrix A is called a lower triangular matrix.

Example:1 $A=\begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 0 \\ 5 & 3 & 6 \end{pmatrix}$ is a lower triangular matrix of order 3.

Example:2 $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 2 & 3 \end{pmatrix}$ is a lower triangular matrix of order 3.

5.3. Operations on Matrices

There are several operations that can be performed on matrices. They are described below.

1. **Addition of Matrices**

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are two matrices of the same order, then their sum A+B is a matrix of order m×n obtained by adding the corresponding elements of A and B.

Thus, $A+B=[a_{ij}]_{m\times n}+[b_{ij}]_{m\times n}=[a_{ij}+b_{ij}]_{m\times n}$

Sum of two matrices A + B exists only when A and B are of the same order.

Example:

(i) Let $A=\begin{pmatrix}6 & 2 & -2\\ 3 & 2 & -2\end{pmatrix}$ and $B=\begin{pmatrix}4 & -3 & -6\\ -1 & 1 & 4\end{pmatrix}$

$A+B=\begin{pmatrix}6+4 & 2-3 & -2-6\\ 3-1 & 2+1 & -2+4\end{pmatrix}$

$=\begin{pmatrix}10 & -1 & -8\\ 2 & 3 & 2\end{pmatrix}$

(ii) Let $A=\begin{pmatrix}1 & 2\\ 4 & 0\end{pmatrix}$ and $B=\begin{pmatrix}2 & 3 & -6\\ -4 & 0 & 9\end{pmatrix}$

Matrix A is of order 2×2.

Matrix B is of order 2×3.

Hence A+B is not defined.

2. **Subtraction of matrices**

If A and B are two matrices of the same order, then their difference is given by A-B=A+(-B),where the matrix (-B) is the negative of the matrix B.

Example: $A=\begin{pmatrix}3 & 4\\ 2 & 2\end{pmatrix}$, $B=\begin{pmatrix}1 & 3\\ -2 & 5\end{pmatrix}$

$(-B)=\begin{pmatrix}-1 & -3\\ 2 & -5\end{pmatrix}$

Then

$$A-B=A+(-B)=\begin{pmatrix}3 & 4\\ 2 & 2\end{pmatrix}+\begin{pmatrix}-1 & -3\\ 2 & -5\end{pmatrix}$$

$$= \begin{pmatrix} 3+(-1) & 4+(-3) \\ 2+2 & 2+(-5) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$$

3. Scalar multiple of a matrix

Let A=[a_{ij}] be an m×n matrix and a c be a scalar (any number c).

Then cA = [ca_{ij}] obtained by multiplying each entry in A by c is called scalar multiple of A by c.

$$A = \begin{pmatrix} 2 & -1 \\ 0 & 9 \\ 9 & -4 \end{pmatrix}$$

$$-A = \begin{pmatrix} -2 & 1 \\ 0 & -9 \\ -9 & 4 \end{pmatrix}$$

Also $$5A = \begin{pmatrix} 5 \text{ x } 2 & 5 \text{ x } (-1) \\ 5 \text{ x } 0 & 5 \text{ x } 9 \\ 5 \text{ x } 9 & 5 \text{ x } (-4) \end{pmatrix}$$

$$= \begin{pmatrix} 10 & -5 \\ 0 & 45 \\ 45 & -20 \end{pmatrix}$$

$$0A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

4. Multiplication of matrices

Let A be an m×n matrix and B be an n×p matrix.

If the number of columns of A is equal to the number of rows of B, then the multiplication of matrices AB is possible.

To obtain the $(i,j)^{th}$ element of matrix AB ,multiply the i^{th} row elements of matrix A by the j^{th} column elements of matrix B.

The i^{th} row of A is [$a_{i1}, a_{i2}, \ldots\ldots a_{in}$] and the j^{th} column entries of B are

b_{1j} , b_{2j} , ……, b_{nj} .

If the product matrix AB is C, then

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{inbnj}$$

$$=\sum_{k=1}^{n} a_{ik}\, b_{kj}\,, 1 \le i \le m, 1 \le j \le p$$

4.4. Related Matrices

1. Transpose of a Matrix

If A is an m×n matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A.

Transpose of the matrix A is denoted by A^T or A'.

Example:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{pmatrix}$$

$$\text{Then } A^T = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{pmatrix}$$

Note :$(A^T)^T=A$

2. Symmetric and Skew-Symmetric matrices

A square matrix A is said to be **symmetric** if $A^T= A$.

A square matrix A is said to be **skew-symmetric** if $A^T=-A$(or $A=-A^T$).

Thus a square matrix $A=[aij]_{n\times n}$ is said to be

Symmetric if $a_{ij}=a_{ji}\ \forall\ i\ and\ j$ and

Skew-symmetric if $a_{ij}= -a_{ji}\ \forall\ i\ and\ j$

Example: The matrix $A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$ is a symmetric matrix.

The matrix $A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ is a skew-symmetric matrix.

Note :

(i) If A is symmetric matrix of order n, then the number of independent elements= $\frac{1}{2}n(n+1)$.

(ii) If A is a Skew-Symmetric matrix of order n, then the number of independent elements= $\frac{1}{2}n(n-1)$.

3. Complex Matrix

If each or a few elements of a matrix are complex numbers, then the matrix is called a complex matrix.

A *complex matrix* can be expressed in the form X+iY, where X and Y are real matrices.

Example: $A = \begin{pmatrix} 2+5i & 1 \\ 3-2i & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} + i\begin{pmatrix} 5 & 0 \\ -2 & 0 \end{pmatrix} = X + iY$ is a complex matrix.

4. Conjugate Matrix

If A=[aij]$_{m\times n}$, then the matrix obtained by replacing each element of A by its complex conjugate is called the *conjugate matrix* of A and is denoted by Ā.

Example:

$If\ A = \begin{pmatrix} 1-i & 4 & 1-i \\ 2+i & -1-i & 2 \end{pmatrix}$,

then Ā=$\begin{pmatrix} 1+i & 4 & 1+i \\ 2-i & -1+i & 2 \end{pmatrix}$

Note :(Ā)=A.

Example:

If $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{pmatrix} and\ B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ show that $(AB)^T = B^T A^T$.

Solution:

$$AB = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

AB=

$$\begin{pmatrix} 1\times1+2\times2+(-1)\times0 & 1\times0+2\times1+(-1)\times1 & 1\times0+2\times0+(-1)\times3 \\ 3\times1+0\times2+2\times0 & 3\times0+0\times1+2\times1 & 3\times0+0\times0+2\times3 \\ 4\times1+5\times2+0\times0 & 4\times0+5\times1+0\times1 & 4\times0+5\times0+0\times3 \end{pmatrix}$$

$$= \begin{pmatrix} 1+4+0 & 0+2+(-1) & 0+0+(-3) \\ 3+0+0 & 0+0+2 & 0+0+6 \\ 4+10+0 & 0+5+0 & 0+0+0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 5 & 1 & -3 \\ 3 & 2 & 6 \\ 14 & 5 & 0 \end{pmatrix}$$

$$(AB)^T = \begin{pmatrix} 5 & 3 & 14 \\ 1 & 2 & 5 \\ -3 & 6 & 0 \end{pmatrix} \dots\dots\dots\dots\dots.(1)$$

$$B^T = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{pmatrix}$$

$$B^T A^T = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 1+4+0 & 3+0+0 & 4+10+0 \\ 0+2-1 & 0+0+2 & 0+5+0 \\ 0+0-3 & 0+0+6 & 0+0+0 \end{pmatrix}$$

$$=\begin{pmatrix} 5 & 3 & 14 \\ 1 & 2 & 5 \\ 4 & 5 & 0 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots.. (2)$$

From (1) and (2) we say that $(AB)^T = B^T A^T$

4.5. Determinant of a matrix

Determinant of a square matrix A may be denoted by det A or |A| or Δ.Determinant of a square matrix A of order 1.ie.,A=[a]

Det A=|A|=a

Determinant of a 2×2 matrix(i.e., of order 2)

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$|A|=a_{11}a_{22}-a_{21}a_{12}$

Minor and co-factor

Let $A=[a_{ij}]_{m\times n}$ be a matrix of order mxn.

The minor of an element a_{ij} is the determinant formed by deleting the i^{th} row and the j^{th} column in which the element a_{ij} exists.

The **minor** of the element a_{ij} is denoted by $\mathbf{M_{ij}}$.

The minor of the element of a determinant of n is a determinant of order(n-1).

The **cofactor** of an element a_{ij} is denoted by $\mathbf{A_{ij}}$ is defined as

$A_{ij}=(-1)^{i+j}M_{ij}$.

Example:

Let $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$|A| = \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The minor and co-factor of the elements a_{11}, a_{22} and a_{32} of the determinant |A| are

M_{11}= Minor of a_{11} = $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$=a_{22}a_{33}-a_{23}a_{32}$

A_{11} = Co-factor of a_{11} $=(-1)^{1+1} M_{11}$

$= + (a_{22}a_{33}-a_{23}a_{32})$

M_{22}= Minor of a_{22}) $=\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

$=a_{11}a_{33}-a_{13}a_{31}$

A_{22} = Co-factor of a_{22} $= (-1)^{2+2} M_{22}$

$=+(a_{11}a_{33}-a_{13}a_{31})$

M_{32} = Minor of a_{32} $=\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

$=a_{11}a_{23}-a_{13}a_{21}$

A_{32} = Co-factor of a_{32} $=(-1)^{3+2} M_{32}$

$=-(a_{11}a_{23}-a_{13}a_{21})$

Expansion of the Determinant

The determinant(Δ)of a matrix A can be expressed as the sum of the products of elements of any row(or column) by their corresponding co-factors.

$\Delta = a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13}$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\Delta = \sum_{j=1}^{3} (-1)^{i+j} \ aijMij$$

$$= \sum_{j=1}^{3} aij\{(-1)^{i+j} \ Mij\}$$

$= \sum_{j=1}^{3} \quad aijAij$

$= a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3}$

for either i=1 or i=2 or i=3.

i.e., the determinant(Δ) is expanded along i^{th} row.

Difference between a matrix and a Determinant

A matrix is an arrangement of numbers in which the number of rows may not be equal to the number of columns.

A matrix defines the representation without any fixed numerical value. However a determinant has a fixed value.

Example 1.

Find the determinant of matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Solution :

Let A=$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$|A| = 1(45-48)-2(36-42)+3(32-35)$

$= 1(-3)-2(-6)+3(-3)$

$= -3+12-9$

$= 0$

$\therefore$ A is a singular matrix.

Example 2.

Find the determinant of the matrix $\begin{pmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$

Solution :

Let $A = \begin{pmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$

$|A| = 2(12-10)+1(-30+25)+1(30-30)$

$= 2(2)+1(-5)$

$= 4-5$

$= -1 \neq 0$

$\therefore$ A is a non − singular matrix.

Example:

Prove that $A^3-4A^2-3A+11I=0$ where A is given by

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix}$$

And I is the unit matrix of order 3.

Solution:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1+6+2 & 3+0+4 & 2-3+6 \\ 2+0-1 & 6+0-2 & 4+0-3 \\ 1+4+3 & 3+0+6 & 2-2+9 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{pmatrix}$$

$$A^3 = A^2 \times A \quad = \begin{pmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{pmatrix}\begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 9+14+5 & 27+0+10 & 18-7+15 \\ 1+8+1 & 3+0+2 & 2-4+3 \\ 8+18+9 & 24+0+18 & 16-9+27 \end{pmatrix}$$

$$= \begin{pmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{pmatrix}$$

A^3-4A^2-3A+11I

$$= \begin{pmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{pmatrix} - 4\begin{pmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{pmatrix} - 3\begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix} + 11\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{pmatrix} - \begin{pmatrix} 36 & 28 & 20 \\ 4 & 16 & 4 \\ 32 & 36 & 36 \end{pmatrix} - \begin{pmatrix} 3 & 9 & 6 \\ 6 & 0 & -3 \\ 3 & 6 & 9 \end{pmatrix} + \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}$$

$$= \begin{pmatrix} 28-36-3+11 & 37-28-9+0 & 26-20-6+0 \\ 10-4-6+0 & 5-16-0+11 & 1-4+3+0 \\ 35-32-3+0 & 42-36-6+0 & 34-36-9+11 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

=0

Hence $A^3-4A^2-3A+11\,I = 0$

Example :

Show that the matrix $A=\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ satisfies the equation $A^2-4A+I=0$ and hence find A^{-1}.

Solution:

$$A=\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

$$A^2=A.A=\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

$$=\begin{pmatrix} 4+3 & 6+6 \\ 2+2 & 3+4 \end{pmatrix}$$

$$=\begin{pmatrix} 7 & 12 \\ 4 & 7 \end{pmatrix}$$

$$-4A=-4\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

$$-4A=\begin{pmatrix} -8 & -12 \\ -4 & -8 \end{pmatrix}$$

$$A^2-4A+I =\begin{pmatrix} 7 & 12 \\ 4 & 7 \end{pmatrix}+\begin{pmatrix} -8 & -12 \\ -4 & -8 \end{pmatrix}+\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 7-8+1 & 12-12+0 \\ 4-4+0 & 7-8+1 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence $A^2- 4A+I = 0$

$A^2-4A=-I$

$A\times A-4\times A=-I$

Post multiplying by A^{-1}, we get

$A \times A \times A^{-1} - 4 \times A \times A^{-1} = -I A^{-1}$

$A \times I - 4I = -A^{-1}$

$A - 4I = -A^{-1}$

$A^{-1} = -A + 4I$

$= -\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} -2 & -3 \\ -1 & -2 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

$= \begin{pmatrix} -2+4 & -3+0 \\ -1+0 & -2+4 \end{pmatrix}$

$= \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

$\therefore A^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

4.6. Typical Square Matrices

1. Orthogonal Matrix

A square matrix A is said to be orthogonal if $AA^T = A^TA = I$

Note :

(i) A^T is an orthogonal matrix.

(ii) $|A| = \pm 1$.

2. Unitary Matrix

A square matrix A is said to be a unitary matrix if $AA^* = A^*A = I$.

Note :

If A is a unitary matrix, then A^T and A^{-1} are unitary matrices.

3. **Involutory Matrix**
   A square matrix A is calledan **involutory matrix**if $A^2 = I$.

4. **Idempotent Matrix**
   A square matrix A is known as Idempotent if $A^2 = A$.

5. **Nilpotent Matrix**
   A square matrix A is known as a **nilpotent matrix** if $A^n = 0$ for some least positive integer n.
   Integer n is called the **index** or **order** of the nilpotent matrix A.

Example:

Show that $\begin{pmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{pmatrix}$ is orthogonal. Determine the value of |A|.

Solution:

Let $A = \begin{pmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{pmatrix}$

$A^T = \begin{pmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{pmatrix}$

$AA^T = \begin{pmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{pmatrix} \begin{pmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{pmatrix}$

$= \begin{pmatrix} cos^2\theta + sin^2\theta & 0 & -sin\theta\ cos\theta + sin\theta\ cos\theta \\ 0 & 1 & 0 \\ -sin\theta\ cos\theta + 0 + cos\theta\ sin\theta & 0 & sin^2\theta + cos^2\theta \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= I$

$AA^T = I$

∴ A is an orthogonal matrix.

$$|A|=\begin{pmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{pmatrix}$$

$$= 1\begin{vmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{vmatrix}$$

$$= cos^2\theta - (-sin^2\theta)$$

$$= cos^2\theta + (sin^2\theta)$$

$$=1$$

$$\therefore\ |A| = 1$$

Example:

Show that the matrix $A=\begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$ is involutory.

Solution :

$$A=\begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$$

$$A^2=A.A=\begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$$

$$=\begin{pmatrix} 25-24+0 & 40-40+0 & 0-0-0 \\ -15+15+0 & -24+25+0 & 0+0-0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$A^2=I$

∴ A is aninvolutorymatrix.

Example :

Show that the matrix A=$\begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$ is Idempotent.

Solution:

A=$\begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$

A^2 = A.A=$\begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} * \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$

=$\begin{pmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{pmatrix}$

= $\begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$

= A

A^2 = A

∴ A is an Idempotent matrix.

4.7. ADJOINT AND INVERSE OF A MATRIX

To describe adjoint and inverse of a matrix, the following definitions are necessary.

1. Singular and Non-singular matrix:

A matrix A=$[aij]_{n\times n}$ is said to be **non-singular** if $|A| \neq 0$.

A matrix A=$[aij]_{n\times n}$ is said to be **singular** if |A|=0.

Example:

Let A = $\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$

|A| = $\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} = 0$

Hence, A is a singular matrix.

Let A = $\begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix}$

|A| = $\begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix}$= 6 – 3 = 3 $\neq 0$

Hence, A is a non-singular matrix.

2. **Adjoint of a Square Matrix**

adj A = transpose of the cofactor matrix

Properties of Adjoint of a matrix

1.A(adj A)=|A|I=adj(A) A, if |A|≠0

2.A$\left\{\frac{(adj\ A)}{|A|}\right\} = I = \left\{\frac{(adj\ A)}{|A|}\right\}A, iff$ |A| ≠0

3. adj(AB) = (adj A) (adj B), iff$|A| \neq 0, |B| \neq 0$

3. **Inverse of a Matrix**

If for a square matrix A, there exists another square matrix B such that AB = BA = I, then B is called the inverse of A and it is denoted by A^{-1}.

$$A^{-1} = \frac{(adj\ A)}{|A|},\ |A| \neq 0.$$

where adjA=transpose of the co-factor matrix.

Note :

1. $A\,A^{-1} = A^{-1}\,A = I$
2. Rectangular matrices cannot have an inverse matrix.

Example:

Find the adjoint of $\begin{pmatrix} 4 & 2 \\ -1 & 3 \end{pmatrix}$

Solution:

Let $A=\begin{pmatrix} 4 & 2 \\ -1 & 3 \end{pmatrix}$

$A_{11} = 3$
$A_{12} = -(-1)=1$
$A_{21} = -2$
$A_{22} = 4$

Cofactor matrix $=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

$=\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}$

adj A =transpose of cofactor matrix

$=\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}$

$=\begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix}$

Example

Find the adjoint of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & 5 \\ 6 & 1 & 0 \end{pmatrix}$

Solution:

Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & 5 \\ 6 & 1 & 0 \end{pmatrix}$

Cofactors of elements are given by

$$A_{11} = \begin{vmatrix} -4 & 5 \\ 1 & 0 \end{vmatrix} = 0 - 5 = -5$$

$$A_{12} = -\begin{vmatrix} 2 & 5 \\ 6 & 0 \end{vmatrix} = -(0 - 30) = 30$$

$$A_{13} = \begin{vmatrix} 2 & -4 \\ 6 & 1 \end{vmatrix} = 2 - (-24) = 26$$

$$A_{21} = -\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} = -(0 - 3) = 3$$

$$A_{22} = \begin{vmatrix} 1 & 3 \\ 6 & 0 \end{vmatrix} = (0 - 18) = -18$$

$$A_{23} = -\begin{vmatrix} 1 & 2 \\ 6 & 1 \end{vmatrix} = -(1 - 12) = -(-11) = 11$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ -4 & 5 \end{vmatrix} = 10 - (—12) = 10 + 12 = 22$$

$$A_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = -(5 - 6) = -(-1) = 1$$

$$A_{33} = \begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix} = -4 - 4 = -8$$

Cofactor Matrix $= \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

$= \begin{pmatrix} -5 & 30 & 26 \\ 3 & -18 & 11 \\ 22 & 1 & -8 \end{pmatrix}$

adj A=transpose of cofactor matrix $\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

$$=\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

$$=\begin{pmatrix} -5 & 3 & 22 \\ 30 & -18 & 1 \\ 26 & 11 & -8 \end{pmatrix}$$

Example :Find the adjoint of the matrix $\begin{pmatrix} cos\alpha & -sin\alpha & 0 \\ sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solution :

Let A=$\begin{pmatrix} cos\alpha & -sin\alpha & 0 \\ sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Cofactor of the elements are

$$A_{11} = \begin{vmatrix} cos\alpha & 0 \\ 0 & 1 \end{vmatrix} = \cos\alpha$$

$$A_{12} = -\begin{vmatrix} sin\ \alpha & 0 \\ 0 & 1 \end{vmatrix} = -\sin\alpha$$

$$A_{13} = \begin{vmatrix} sin\alpha & cos\alpha \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{21} = -\begin{vmatrix} -sin\ \alpha & 0 \\ 0 & 1 \end{vmatrix} = -(-\sin\alpha) = sin\alpha$$

$$A_{22} = \begin{vmatrix} cos\ \alpha & 0 \\ 0 & 1 \end{vmatrix} = \cos\alpha$$

$$A_{23} = -\begin{vmatrix} cos\ \alpha & -sin\alpha \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{31} = \begin{vmatrix} -sin\ \alpha & 0 \\ cos\ \alpha & 0 \end{vmatrix} = 0$$

$$A_{32} = -\begin{vmatrix} cos\ \alpha & 0 \\ \sin\alpha & 0 \end{vmatrix} = 0$$

$$A_{33} = \begin{vmatrix} cos\ \alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{vmatrix} = cos^2\alpha + sin^2\alpha = 1$$

Cofactor matrix A_{ij}=$\begin{pmatrix} cos\alpha & -sin\alpha & 0 \\ sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$

AdjA =$(A_{ij})^T$

= =$\begin{pmatrix} cos\alpha & sin\alpha & 0 \\ -sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Example:

If A=$\begin{pmatrix} 2 & 3 \\ 4 & 8 \end{pmatrix}$ verify that A(adj A)=(adj A)A=det(A)I

Solution:

Let A=$\begin{pmatrix} 2 & 3 \\ 4 & 8 \end{pmatrix}$

$|A| \quad =\begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix}$

= 16-12

= 4

$|A|=4$

$A_{11} = 8$
$A_{12} = -4$
$A_{21} = -3$
$A_{22} = 2$

Cofactor matrix = $\begin{pmatrix} 8 & -4 \\ -3 & 2 \end{pmatrix}$A

adj A $= transpose\ of\ cofactor\ matrix$

$= \begin{pmatrix} 8 & -3 \\ -4 & 2 \end{pmatrix}$

A x (adj A)=$\begin{pmatrix} 2 & 3 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} 8 & -3 \\ -4 & 2 \end{pmatrix}$

$= \begin{pmatrix} 16-12 & -6+6 \\ 32-32 & -12+16 \end{pmatrix}$

$$=\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

$$= 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= 4\ I$$

(adj A) x A $=\begin{pmatrix} 8 & -3 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 8 \end{pmatrix}$

$$= \begin{pmatrix} 16-12 & 24-24 \\ -8+8 & -12+16 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

$$= 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= 4\ I$$

$\therefore A(\text{adj } A) = \text{adj } A(A) = (\det A)\ I$

Hence it proved

Example :

Find the inverse of the matrix $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$

Solution :

Let A = $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$

$|A| = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$

$= 2 - 0$

$= 2$

$|A|=2$

$|A|\neq 0$

Hence A is a non-singular matrix.

$\therefore\ A^{-1}$ exists.

Cofactors of the elements of A are

$A_{11} = 1$

$A_{12} = 0$

$A_{21} = -1$

$A_{22} = 2$

Cofactor matrix $A_{ij} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$

adj A= $(A_{ij})^T$

adj A= $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$

$$A^{-1} = \frac{1}{|A|} adj\ A$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

Example :

Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$

$|A| \quad = 1(-12-12)-1(-4-6)+3(-4+6)$

$=1(-24)-1(-10)+3(2)$

$=-24+10+6$

$=-8$

$|A| \neq 0$

$\therefore$ A is a non $-$ singular matrix.

$\therefore A^{-1}$ exists.

Cofactors of elements of |A| are

$A_{11}=\begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} = -12-12=-24$

$A_{12}=-\begin{vmatrix} 1 & -3 \\ -2 & -4 \end{vmatrix} = -(-4-6)=10$

$A_{13}=\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} = -4+6=2$

$A_{21}=-\begin{vmatrix} 1 & 3 \\ -4 & -4 \end{vmatrix} = -(-4+12)=-8$

$A_{22}=\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} = -4+6=2$

$A_{23}=-\begin{vmatrix} 1 & 1 \\ -2 & -4 \end{vmatrix} = -(-4+2)=2$

$A_{31}=\begin{vmatrix} 1 & 3 \\ 3 & -3 \end{vmatrix} = -3-9=-12$

$A_{32}=-\begin{vmatrix} 1 & 3 \\ 1 & -3 \end{vmatrix} = -(-3-3)=6$

$A_{33}=\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 3-2=1$

Cofactor Matrix $A_{ij} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

$= \begin{pmatrix} -24 & 10 & 2 \\ -8 & 2 & 2 \\ -12 & 6 & 2 \end{pmatrix}$

adj $A=(A_{ij})^T$

$$= \begin{pmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} . \text{adj } A$$

$$= \frac{1}{-8} \begin{pmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{pmatrix}$$

Example :

Find the inverse of the matrix $A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -3 & 6 \\ 1 & 1 & 7 \end{pmatrix}$

Solution :

Let $A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -3 & 6 \\ 1 & 1 & 7 \end{pmatrix}$

$|A| = 1(-21-6) + 2(14-6) + 2(2+3)$

$= 1(-27) + 2(8) + 2(5)$

$= -27 + 16 + 10$

$= -1$

$|A| \neq 0$

$\therefore$ A is a non − singular matrix.

$\therefore$ A^{-1} exists.

Cofactors of elements of |A| are

$A_{11} = \begin{vmatrix} -3 & 6 \\ 1 & 7 \end{vmatrix} = (-21 - 6) = -27$

$A_{12} = -\begin{vmatrix} 2 & 6 \\ 1 & 7 \end{vmatrix} = -(14 - 6) = -8$

$A_{13} = \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = 2 + 3 = 5$

$A_{21} = -\begin{vmatrix} -2 & 2 \\ 1 & 7 \end{vmatrix} = -(-14 - 2) = 16$

$A_{22} = \begin{vmatrix} 1 & 2 \\ 1 & 7 \end{vmatrix} = 7 - 2 = 5$

$A_{23} = -\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = -(1 + 2) = -3$

$A_{31} = \begin{vmatrix} -2 & 2 \\ -3 & 6 \end{vmatrix} = (-12 + 6) = -6$

$A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} = -(6 - 4) = -2$

$A_{33} = \begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} = (-3 + 4) = 1$

Cofactor Matrix $A_{ij} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

$$= \begin{pmatrix} -27 & -8 & 5 \\ 16 & 5 & -3 \\ -6 & -2 & 1 \end{pmatrix}$$

adj A = $(A_{ij})^T$

$$= \begin{pmatrix} -27 & 16 & -6 \\ -8 & 5 & -2 \\ 5 & -3 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} . adj\ A$$

$$= \frac{1}{-1}\begin{pmatrix} -27 & 16 & -6 \\ -8 & 5 & -2 \\ 5 & -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 27 & -16 & 6 \\ 8 & -5 & 2 \\ -5 & 3 & 1 \end{pmatrix}$$

Example :

Find the inverse of the matrix $\begin{pmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{pmatrix}$

Solution :

Let B=$\begin{pmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{pmatrix}$

|B|=1(-30+13)-3(-6+3)-4(13-15)

=1(-17)-3(-3)-4(-2)

=-17+9+8

|B|=0.

B is a singular matrix

∴ B^{-1} does not exists.

Properties of inverse of a matrix

1. The necessary and sufficient condition for a square matrix A to possers the inverse is that A is to be non-singular

   i.e$|A| \neq 0$

2. The inverse of a matrix, if exists is unique.

3. If A and B are two non-singular matrices of the same order, then $(AB)^{-1}=B^{-1}A^{-1}$. i.e, inverse of a product of two matrices is the product of their inverses in the reverse order.

4. If A is non – singular, then

   (a) $(A^{-1})^{-1}$= A

(b) $(A^{-1})^T = (AT)^{-1}$

5. A^{-1} is an orthogonal matrix.

UNIT V

GRAPH THEORY

5.1 GRAPHS AND BASIC TERMINOLOGIES

A graph is a mathematical concept which can be used to model many concepts from –the real world.

A **graph** consists of a pair of sets, represented as *G* = (V, *E*), where V is a non-empty set of **vertices** (also called **nodes**) and E is a set of **edges** (sometimes called **arcs**).

An edge can be represented as a pair of nodes (u,v) indicating an edge from node u to node v.

Two vertices/nodes *x* and y of *G* are *connected* if there is an edge *xy* between them, and these vertices are then called *adjacent* or *neighbour* vertices/nodes. Here, the nodes *x* and y are called the *endpoints* of the edge.

x  y

In a graph *G*, a node which is not adjacent to any other node is called an *isolated node*.

A graph is *finite* if it has a finite number of vertices and a finite number of edges, otherwise it is infinite.

If G is finite, *G(*V)*denotes* the number of vertices in *G* and it is called the *order* of *G.*

Similarly, *E(G)* denotes the number of edges in *G* and it is called the *size* of G.

The graph shown in figure 5.1has four vertices a,b, c and d.

(a,b) is a pair of vertices which are connected, and this connectivity represents an edge between them. Now a and*b* are the *end points* of the edge (*a,* b).

*N*eighbours of vertex *a* in this graph are *b* and *c* as there are edges from *a* to *b* and *a* to *c.*

Vertex *d* is the *isolated* vertex, as it is not adjacent to any other vertices.

It is an example of finite graph and its Order of G is 4 as V = {a, b, c, d}.

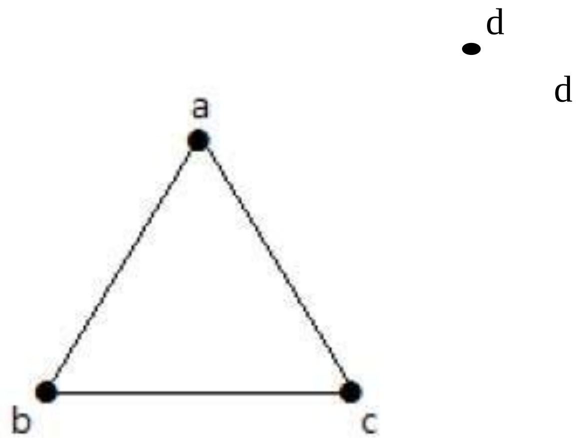


Figure 5.1 A graph with isolated vertex

1. Undirected and Directed Graphs

Graphs may be *directed* or *undirected*.

A graph is directed (or *digraph*) when direction of edge from one vertex to another is defined, otherwise it is an undirected graph.

Undirected edge between vertices *u* and v is expressed as *(u,* v).

Directed edge between vertices *u* and *v* is expressed as <*u,v*>

A simple directed graph is shown in Figure 5.2

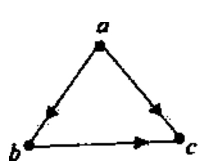


Figure 5.2 Simple Directed Graph

2. Weighted Graph

A weighted graph associates a value (*weight*) with every edge in the graph.

In other words, when a weight (may be cost, distance, etc) is associated with each edge of a graph, then it is called as weighted graph, otherwise **unweighted** graph.

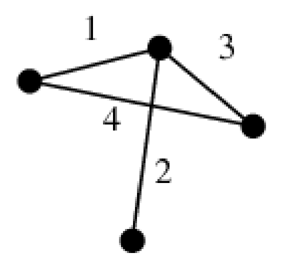


Figure 5.3 *edge-labeled graph*

3. Self-Edge or self –Loop

In graph theory, a **loop** (also called a **self-loop** or a "buckle") is an edge that connects a vertex to itself. A simple graph contains no loops.

A graph with self loop is shown Figure 5.4.

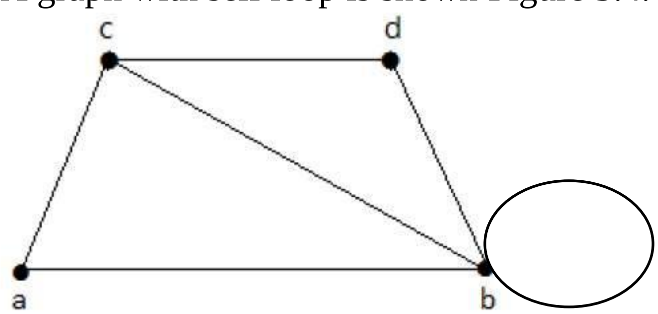
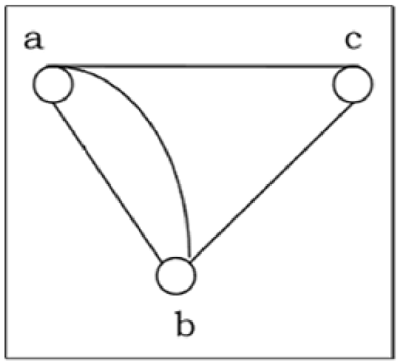


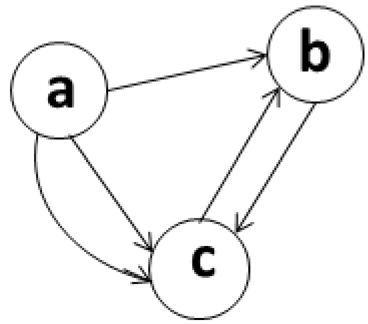
Figure 5.4

4. Multiple or parallel Edges

If a pair of nodes is joined by more than one edge, then such edges are called *multiple* or *parallel* edges.

In *undirected* graph, two edges(v_i, v_j) and $(v_p = v_q)$ are parallel edges if$v_i = v_p$ and $v_j = v_q$.





(a) Undirected graph with parallel edges

(b) Directed graph with parallel edges

Figure 5.5

In the directed graph, the edges between vertices a and b parallel edges. The edge pair between the vertices b and c are not *parallel edges*, since the directions of the edge pair are *opposite.*

5. Path in a Graph

A path in a graph is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence. Clearly, vertices as well as edges may be repeated in a path.

A path from *u* to w is a sequence of edges $(u, v_1), (v_1, v_2) \ldots, (v_{k-1}, w)$connecting *u* with w.

A path may be termed as *walk* also.

The number of edges in a path is termed as its *length.*

For example,*(a, b*), *(b,b*), (*b*, d), *(d, c)*, *(c, b*), (*b*, d) is one path of the graph as shown in Figure 5.4. and its length is 6.

A path with no repeated vertex is called a *simple path.*

In Figure 5.4. (a, *b)*, (*b, d), (d, c) is simple path.*

A path with no repeated edge is termed as *trail*.

In a closed trail, the first and the last vertices are same.

A closed path is a path that starts and ends at the same point, otherwise the path is open. Edge repetition is allowed on the closed path.

In Figure 5.4. (a, *b), (b, d), (d, c),(c,a) is a closed path.*

A **Cycle** (**circuit**/**tour**) is a closed path of non-zero length that does not contain any repeated edges. Vertices other than the *end* (i.e., *start)* vertex may also be repeated.

In Figure 5.4, *(a, b),* (*b, b), (b, d), (d, c),(c, a)* is a cycle.

A *simple cycle* is a cycle that does not have any repeated vertex except the first and the last vertex. *(a, b),(b, d), (d, c),(c, a)* is an example of simple cycle.

A graph without cycles is called *acyclic.*

A *tree* is an acyclic and connected graph.

A *forest* is a set of trees.

6. Connected Graph

A graph is called *connected* if and only if for any pair of nodes *u,* v, there is at least one path between *u* and v. Otherwise, it is disconnectecd.

Clearly, the graph in Figure 5.4 is a connected undirected graph, whereas the graph given in Figure 5.1 is disconnected.

7. Types of Connectivity in Graphs

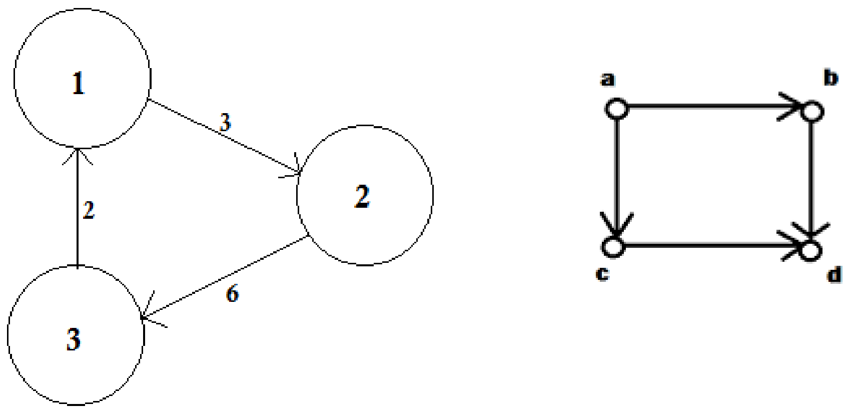
A connected graph must have at least two vertices.

A graph is *strongly connected* if and only if every pair of vertices in the graph are reachable from each other. i.e., if there are paths in both directions between any two vertices.

Otherwise, the graph is of *weakly* or *unilaterally* connected.

The graph in Figure 5.6(a) is an unilaterally connected graph, as it has a path from *a* to *c* but no path exists from *c* to *a,* and so on.

A graph is *strictly weakly* connected if it is not unilaterally connected. Thus, a strictly weakly connected graph may have many *sources* and *sinks* (destinations). The graph given in Figure 5.6(b) is an example of *strictly* weakly connected graph.



(a) (b)

Figure 5.6

8. Simple Graph, Multi – Graph, and Pseudo-Graph

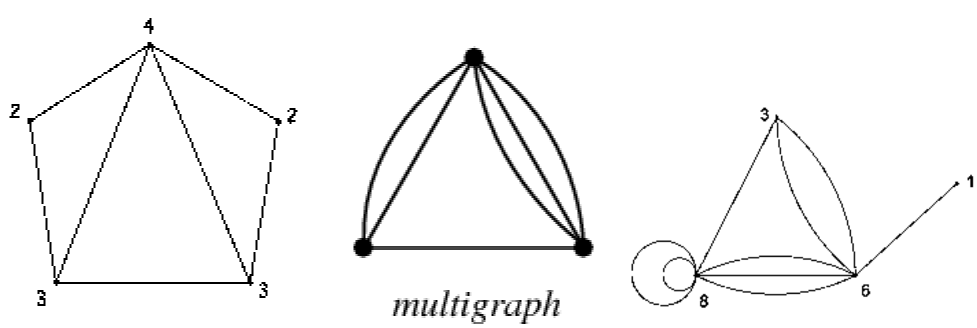
A directed or undirected graph which has neither self-loops nor parallel edges is called *simple graph.*

However, cycle(s) is (are) allowed in a simple graph.

Further, a simple graph may contain *isolated* vertex also.

The graph as shown in Figure 5.7(a) is a simple connected graph, since it has no self loop and parallel edges.

Further, the graph in Figure 5.6 is a *simple directed* graph, as it has no self-loop or parallel edges. On the other graph in Figure 5.1 is a *simple disconnected* graph.



(a) (b) (c)

Figure 5.7

Any graph (directed or undirected) which contains some parallel edges is called a multigraph. In multi-graph, no self-loop is allowed but cycle may be present.

Figure 5.7 (b) is an example of multi-graph, since it has parallel edges but *no* self loop.

A directed or undirected graph in which self-loop(s) and parallel edge(s) are allowed is called a *pseudo-graph.* Figure 5.7 (c) is an example of *pseudo-graph.*

9.Degree of vertex

The degree of a vertex of an undirected graph is the number of edges incident on itcounting self loop twice. The degree of a vertex G is denoted by $deg_G(v)$.

For example In the undirected graph in Figure 5.4, the degree of *a* [i.e., $deg_G(a)$ is 2, degree of b [i.e., $deg_G(b)$] is 5 (since there is a self-loop at *b),* degree of *c* is 3 and degree of d is 2.

In directed graph *G,* we consider two types of degrees of vertices: (a) ***in-degree*** *and (b)* ***out-degree****.*

The in-degree of a vertex v of G, denoted by $deg_G-(v)$, is number of edges moving into that vertex.

The *out-degree* of v, denoted by $deg_G^+(v)$is the the number of edges moving out from that vertex.

The sum of the in-degree and the *out-degree* of a vertex is called the *total degree* of that vertex.

For example:

In the directed graph in figure 5.6(a),the in-degree of 1,$deg_G-(1)$ is 0(zero) and but its out-degree, $deg_G^+(v)$ is 1.

Hence the total degree of 1 is $deg_G-(1) + deg_G^+(1) = 2$.

A vertex with zero in-degree is called a *source vertex* and a vertex with zero out-degree is called *sink* vertex.

A vertex of degree 0(zero) is called isolated vertex.

A vertex is pendant vertex if and only if its degree is 1.

The vertex d in Figure 5.1 is an isolated vertex, as its degree is zero, whereas the vertex 1 of the graph in Figure 5.7(c) is a pendant vertex (because its degree 1).

10. Degree Sequence of a Graph

Let G be a graph with vertices v_1, v_2,v_3, …v_n. The monotonically increasing sequence (d_1, d_2, d_3, …d_n), where $d_i = \deg_G(v_i)$ is called the degree sequence of the graph G.

The degree sequence of the graph in figure 5.4 is (2,2,3,5).

Note :

The degree of a graph G is the *maximum* of the degrees of all nodes in G.

If the number of edges $m = O(n)$ (where n is the number nodes in the graph), then the graph is said to be *sparse*.

If m is larger than linear order of n, i.e., $m = 0(n^2)$ (but as long as there are no multiple edges), then the graph is called *dense*.

Theorem 5.1

A simple graph with n ≥ 2vertices contains atleast two vertices of the same degree.

Proof :

Let *G* be a simple graph with n ≥ 2vertices.

Since G is a simple graph ,it has no loop and parallel edges.

We know that, the degree of a vertex of a simple graph G on n vertices cannot exceed n-1.

So, degree of each vertex is ≤ 1.

Assume that all the vertices of *G* have distinct degrees.

Thus, the degrees, 0, 1, 2, 3,….,n-1,are possible for n vertices of G.

Let u be the vertex with degree 0.Clearly,u is the isolated vertex.

Let v be the vertex with degree n-1,then v must have n-1 adjacent vertices.

In fact, it is possible if the vertex v is adjacent to each vertex of the graph G, it is also adjacent to u. But it is assumed that u is an isolated vertex. i.e., it is not adjacent with any vertex of G.

Hence ,either u is not an isolated vertex or the degree of v is not n-1.

So, contradiction occurs on the assumption of different distinct degrees of vertices of G.

Thus, the contradiction proves that a simple graph contains at least two vertices of same degree.

Note :

The above theorem can be clearly understood by taking some examples.

(i) Suppose, the simple graph G has two vertices v_1 and v_2.

Since it is a simple graph, G has no loop or parallel edges.

First, consider that both the vertices are isolated.

Hence, $deg_G(v_1) = deg_G(v_2) = 0$.

So, both the vertices have same degree 0.

Second, suppose that they are adjacent to each other(but no parallel edges).

Hence, $deg_G(v_1) = deg_G(v_2) = 1$.

So, both the vertices have same degree 1.

(ii) Suppose, the simple graph G has three vertices v_1 , v_2 and v_3.

The graph G is a simple graph, so it has no loop or parallel edges.

First, consider that all the vertices are isolated.

Hence, $deg_G(v_1) = deg_G(v_2) = deg_G(v_3) = 0$.

So, at least two vertices (here all the vertices)have same degree 0.

Second, suppose, one vertex is isolated and the remaining two are adjacent to each other.

Then, degree sequence is(0,1,1) and it means at least two vertices out of three have same degree.

Third, suppose G has no isolated vertices out of three.

Then, degree sequence is (1,1,2) and it means at least two vertices out of three have same degree.

Hence, it is true for a simple graph with any number of vertices.

The above theorem is true for both directed as well as undirected graphs.

Theorem 5.2(The Handshaking theorem)

If G=(V,E) is a graph with e number of edges, then

$$\sum_{v\in V} deg_G(v) = 2e$$

i.e., the sum of degrees of the vertices of G is always even.

For directed graph,

$$\sum_{v\in V} deg_G(v) = \sum_{v\in V} deg^-(v) + \sum_{v\in V} deg^+(v)$$

i.e., the sum of degrees of the vertices is the sum of the in-degrees and the out-degrees of the vertices.

Proof :

Let G be an undirected graph.

The degree of a vertex of G is the number of edges incident with that vertex.

Now, every edge is incident with exactly two vertices.

Hence, each edge gets counted twice, one at each end.

Thus, the sum of the degrees equals twice the number of edges.

Let G be a directed graph.

Then in-degree and out-degree of each vertex of G are considered.

However, the sum of the in-degree and out-degree of a vertex is the total degree of that vertex.

Further, every edge is incident with exactly two vertices. So, here also, each edge gets counted twice: one as in-degree and the other as out-degree.

Thus, the sum of the degrees(in-degrees and out-degrees) of all the vertices equals twice the number of edges.

Note:

(i) The name of this theorem is handshaking because if several people shake hands, the total number of hands involved must be even(since for every handshaking, two hands are required).

(ii) This theorem applies even if multiple edges and self-loops are present in graph.

(iii) The theorem it is true for both connected and disconnected graphs.

(iv) If sum of degrees of the vertices of a graph is given, then the number of edges present in that graph can be computed. But the reverse is not possible.

Corollary :

In a graph, total number of odd-degree vertices is even.

Proof :

Let G=(V,E) be a graph, where K_1 and K_2 are the set of vertices with odd degree and even degree, respectively.

Now, $\sum_{v_i \in V} degG^{(v_i)} = \sum_{v_i \in K_1} degG(v_i) + \sum_{v_i \in K_2} degG(v_i)$

$$2e = \sum_{v_i \in K_1} degG(v_i) + \sum_{v_i \in K_2} degG(v_i)$$

[Since sum of the degree of vertices is twice the number of edges(e), and it is always even.]

Further, sum of the even-degree vertices is even.

i.e., $\sum_{v_i \in K_2} degG(v_i)$ is even.

Clearly,

$\sum_{v_i \in K_1} degG(v_i)$ is even.

i.e., the sum of the odd-degree vertices is also even, Again,$\sum_{v_i \in K_1} degG(v_i)$ is even only if number of vertices of K_1 is even.

Hence, the number of odd degree vertices is even.

[For example, suppose three vertices contain odd degrees 1,3,5respectively.Clearly,there sum may not be even, since number of vertices is 3 which is an odd number.]

Note:

The sum of two numbers(say,n_1 and n_2)gives even if both of n_1 and n_2 are either odd or even. i.e., odd+odd=even, even+even=even.

Theorem 5.3

If G=(V,E) be a directed graph with e number of edges, then

$\sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v)$

i.e., the sum of the out-degrees of the vertices of G equals the sum of the in-degree of the vertices, which equals the number of edges in G.

Proof :

Any directed edge of G contributes 1 out-degree and 1 in-degree. Also, a self-loop contributes two degrees(1 out-degree and 1 in-degree).

Hence, the theorem is proved.

Example:

In the directed graph in Figure 5.6 (a),the in-degree of1, $degG^{-}(1)$ is 1 and its out-degree $degG^{+}(1)$ is 1.

Hence, the total degree of 1 is 2.

$$degG(1) = de\,gG^{-}(1) + de\,gG^{+}(1) = 1 + 1 = 2,$$

$$degG(2) = degG - (2) + de\,gG^{+}(2) = 1 + 1 = 2,$$

$$degG(3) = de\,gG^{-}(3) + de\,gG^{+}(3) = 1 + 1 = 2.$$

Hence, $de\ gG^{-}(1) + de\ gG^{-}(2) + de\ gG^{-}(3) = 1 + 1 + 1 = 3 =$ e(number of edges) and

$de\ gG^{+}(1) + de\ gG^{+}(2) + de\ gG^{+}(3)$=1+1+1=3=e(number of edges).

Example 5.1:

Show that the degree of a vertex of a simple graph G on n vertices cannot exceed n-1.

Solution :

Let v be a vertex of G.

Since G is simple, no multiple edges or self-loops are allowed in G.

Thus v can be adjacent to at most all the remaining n-1 vertices of G.

Hence, v may have maximum degree n-1 in G.

If the degree of v becomes more than(n-1),then there must have self-loops or parallel edges in the graph, which is not allowed in simple graph.

So, the degree of a vertex $v \in V(G)$ in a simple graph lies in the range $0 \leq\ degG(v) \leq n - 1$.

In particular, it is 0 if the vertex is isolated.

Note :

The above inequality is true for both directed and undirected simple graphs.

Example 5.2 :

Show that the maximum number of edges in a simple undirected graph with n vertices is n(n-1)/2.

Solution :

By the handshaking theorem, we know

$$\sum_{v \in V} deg_G(v\ \ ) = 2e$$

where e is the number of edges with n vertices in the graph G.

This implies

$d(v_1) + d(v_2) + d(v_3) + \cdots + d(v_n) = 2e$ ……. (5.1)

Since maximum degree of each vertex in a simple graph can be (n-1).

Therefore, Eq. (5.1) can be written as

(n-1)+(n-1)+…+ up to n terms(considering maximum degree for each vertex)

=n(n-1)

=2e

Hence,

e(maximum number of edges in a simple graph with n vertices)=n(n-1)/2.

Note :

Maximum number of edges in a simple directed graph G is 2n(n-1)/2 =n(n-1),since in a simple directed graph, edges with opposite direction between any pair of vertices are allowed.

Example 5.3 :

For a simple graph with n vertices, what is the minimum number of edges required to ensure that the graph is connected?

Solution :

Let S c V be a set of vertices for which each vertex in S has degree 0.

If S has just one vertex(the minimum case of a disconnected graph),then(n-1) edges are possible between S and (V-S).

Therefore, the maximum possible number of edges in a disconnected graph is

$$n(n-1)/2-(n-1) = (n-1)[(n/2)-1]$$
$$= (n-1)(n-2)/2$$

Clearly, the minimum number of edges in a connected graph is

$$= 1+(n-1)(n-2)/2$$
$$=[2 +(n^2-3n+2)]/2.$$
$$= (n^2-3n+4)/2$$

Note : To check the **existence of a graph** when its **degree sequence** is given.

1. If the sum of the degrees of the vertices of the graph is not even, then graph corresponding to the given degree sequence cannot be drawn (application of handshaking theorem).
2. If the total number of odd degree vertices (counted from the given degree sequence) is odd, then graph corresponding to the given degree sequence cannot be drawn.

Hence, for the existence of any graph G, the number of odd-degree vertices must be even, and this point can be applied only for confirmatory checking. i.e., it is not compulsory to consider.

Now, if both the above-mentioned conditions are false(i.e., when the sum of the degrees is even and the number of odd-degree vertices is also even, then it is certainly possible to draw one graph, but it may not be possible to draw a simple graph following the given degree sequence.

For checking the existence of a simple graph, we must concentrate on its properties. Some examples on degree sequence are given below.

Example 5.4 :

Is there a simple graph corresponding to the following degree sequences?

(a) (1,1,2,3)

(b) (2,2,4,4)

Solution :

(a) The total number of odd-degree vertices in a graph is even.

The number of odd-degree vertices is 3.

Hence, no graph corresponding to this degree sequence can be drawn.

Sum of degrees=1+1+2+3=7, which is odd.

By handshaking theorem, the sum of degrees of any simple graph must be even.

Hence, no graph exists for this case.

(b) The sum of the degree of the vertices is 12 which is even.

Also, the number of the odd-degree vertices is 0 which is even.

So, a graph can be drawn, using the given degree sequence.

Now, let us check if any simple graph is possible to draw or not.

The number of vertices is 4.

However, the degree of any vertex in a simple graph G on n vertices cannot exceed n-1, the degree of any vertex cannot be 4.

Hence, no simple graph corresponding to the given degree sequence can be drawn.

Example 5.5 :

Does there exist a simple graph with seven vertices having degrees(1,3,3,4,5,6,6)?

Solution:

The sum of the degrees of the vertices is 1+3+3+4+5+6+6=28 and it is an even number.

Also, the number of odd-degree vertices is even.

So, the graph corresponding to the given degree sequence exists.

Now, let us check whether any simple graph exists or not.

Assume that it exists. Here, two vertices out of seven have degree 6.So,each of these two vertices is adjacent to the rest six vertices of the graph. Accordingly, the degree of each vertex should be at least 2. i.e., it may not be 1.But in the degree sequence, no vertex with degree 2 is provided. Moreover, a vertex with degree 1 is given.

Therefore, we arrive at a contradiction in out assumption. Thus no simple graph, following the given degree sequence, can be drawn.

Example 5.6 :

For the graph G as shown in figure 5.8, write the degree sequence of G.

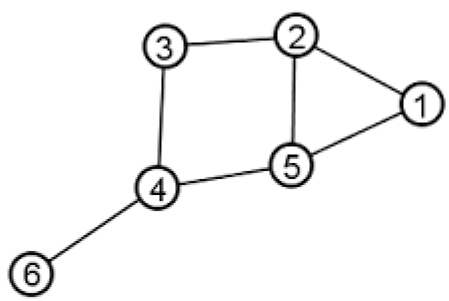


Figure 5.8

Hence, find the number of odd-degree vertices and the number of edges in the graph G.

Solution:

The degree sequence is given as {3,3,3,2,2,1}.

Hence, the number of the odd-degree vertices is 4,which is even as per the corollary of handshaking theorem.

Now, the sum of degrees of all vertices is 2e, where is the number of edges. So, we get

3+3+3+2+2+1=2e

2e =14

e= 7

Hence, the number of edges in the given graph is 7and can be verified by counting.

Example 5.7 :

For each of the following degree sequences, determine if there exists a graph whose degree sequence is given. If possible draw the graph or explain why such a graph does not exist.

i. (1, 1, 1, 1, 1)

ii. (1, 1, 1, 1, 1, 1)

Solution

(i) The given degree sequence is (1, 1, 1, 1, 1).

Sum of the degrees of the vertices =1 + 1 + 1 + 1 + 1 = 5 = odd number.

Hence, it is not possible to draw any *graph corresponding to the degree* sequence (*1*,*1*,*1*, *1*,*1*).

(ii) The given degree sequence is (1, 1, 1, 1, 1, 1).

Sum of degrees = 1 + 1 + 1 + 1 + 1 + 1= 6 = even number

Therefore, e = number of edges = 6/2 = 3.

Here, n (number of vertices in the graph) = *6*.

Also, number of odd-degree vertices *is 6 and it is an even number*.

Hence, the graph corresponding to the given sequence (1,1,1,1,1,1) can be drawn.

Example 5.8 :

Let G be a simple graph with 12 edges. If G has 6 vertices of degree 3 and the rest of the vertices have degree less than 3, then find the (a) minimum number of vertices and (b) maximum number of vertices.

Solution:

Number of edges e = 12

Suppose the total number of vertices in *G* is *p*.

Given that 6 vertices have degree 3.

Hence, the sum of degrees =3 *6 = 18.

The rest *(p* — 6) vertices have degree less than 3. i.e., their degree lies inclusively between 0 and 2.

Here, to find the *minimum* number of vertices, *(p* — 6) vertices must have *maximum* degree [i.e., 2]

Therefore, applying the handshaking theorem, we get

$$18 + 2(p - 6) = 2e$$

$$18 + 2p - 12 = 24$$

$$2p + 6 = 24$$

$$2p = 18$$

$$\Rightarrow p = 9$$

Minimum number of vertices = 9

To calculate the *maximum* number of vertices, *(p* — 6) vertices must have *maximum* degree [i.e., 1]

Sum of degrees = $18 + (p - 6) = 2e = 24$

$$p + 12 = 24$$

$$\Rightarrow p = 12$$

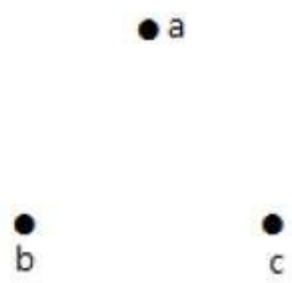
Maximum number of vertices = 12

5.2 TYPES OF GRAPHS

Some important types of graphs are introduced here. These are often used in many applications.

5.2.1 Null Graph

A graph which contains only isolated nodes is called a null graph. i.e the set of edges in a null graph is empty. Null graph on n vertices in denoted by N_nNull graph (N_3) with 3 vertices is shown below



5.2.**2 Complete Graph**

A graph G is said to be complete if every vertex of G is connected with every other vertex of G. i.e., every pair of distinct vertices contains exactly one edge. Complete graph on n vertices is denoted by K_n.

Some complete graphK_1, K_2, K_3, K_4, K_5, K_6, $K_7$are shown below.

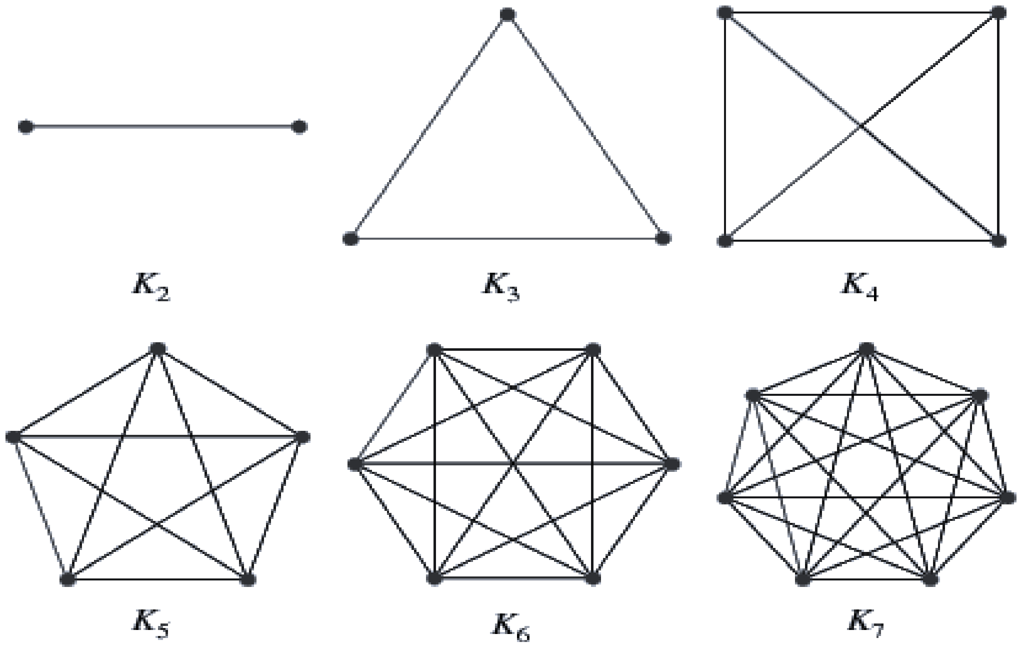


Figure 5.9

A complete graph *G* is a simple graph and it may be *directed* as well *as* undirected. Any complete graph K_n with *n* vertices has exactly $n(n — 1)/2$ edges.

Directed graph K_3 is shown in Figure 5.10.

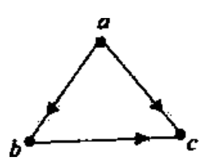


Figure 5.10

5.2.3 Regular Graph

A graph in which all the vertices are of *same degree is called a regular graph. If the degree* each vertex is *r,* then the graph is called a regular graph of degree r, and it is denoted byR_r. A regular graph may be *directed* or *undirected.* When it is directed, *then the degree of each vertex is computed* as *the* sum of its *in-degree* and *out- degree.*

A complete graph is a regular graph of degree n-1 or it is called (n-1) regular graph. Obviously, if a graph is null graph, then it is 0 regular (as degree of each vertex is 0)

2-regular graph with 5 vertices is given in Figure 5.11

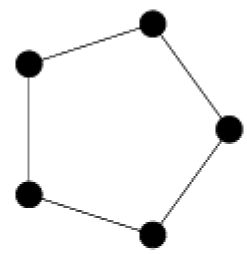


Figure 5.11

Note:

If a graph G with n vertices is r-regular, then the number of edges of G is

r * n/2.

Since the graph has n vertices n vertices are r-regular, the sum of the degree of the vertices is n * r. Also, sum of degrees of a graph equals to twice the number of edges. Hence, the number of the edges of the regular graph with n vertices is r * n/2.

Example 5.9

Find the number of edges of a 4-regular graph with 6 vertices.

Solution :

Here n = 6 and r = 4.

Number of edges $(e) = r * \frac{n}{2} = 4 * \frac{6}{2} = 12$

Example 5.10

Is it possible to draw a 3-regular graph with 5 vertices.

Solution :

Number of vertices n = 5

r= 3

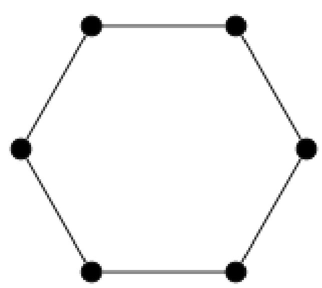
Sum of the degrees of the vertices = 5 * 3 = 15, *which is not divisible* by 2. Therefore, it is not possible to draw a 3-regular with 5 vertices.

Note :A graph with n vertices is r-regular if either r or n or both are even.

Cycles

The cycle C_n , $n \geq 3$, consists of n vertices and n edges so that the second endpoint of the last edge coincides with the staring vertex.

A cycle with 6 vertices is shown below.

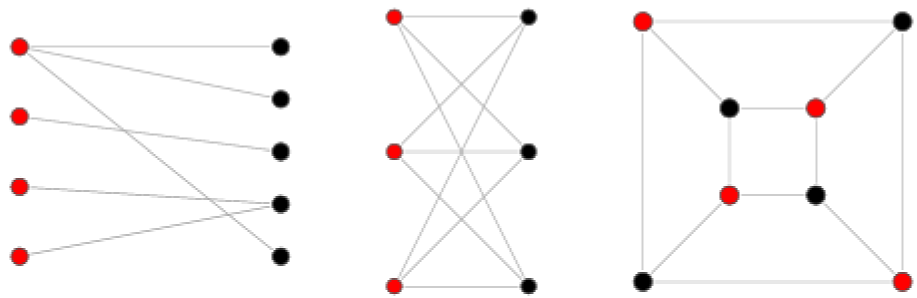


5.2.4 Bipartite Graph

A graph *G* = (V, *E*) is a bipartite graph if the vertex set *V* can be partitioned into two disjoint subsets, say, V_1 and V_2such that every edge in *E* connects a vertex in $V_1$to the vertex in V_2.

But no edge in G connects either of the two vertices in V_1 or two vertices in V_2.

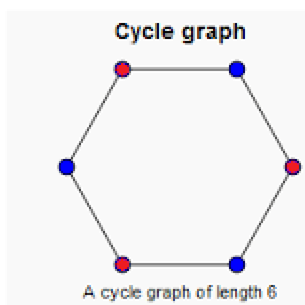
(V_1, *V_2)* is called a *bipartition* of G. Some examples of bipartite graph are shown below.



Example 5.11

Show that the graph C_6 is bipartite.

Solution:



In this graph, the two distinct sets of vertices are shown in distinct colours. Hence, C_6 is bipartite.

Procedure to check whether a graph G is bipartite or not

Step 1 Arbitrarily select a vertex from G and include it into set 1.

Step 2 Consider the edges directly connected to that vertex and put the other end vertices of these edges into set 2.

Step 3 Now, pick up one vertex from set 1and consider the *edges* directly connected to that vertex, and put the other end vertices of these edges into set 2.

Step 4 At each step, step 2 and step 3, check if there is any edge among the vertices of set 1 or set 2.

If so, construction of sets is stopped and the given graph is not bipartite graph, then return.

Else continue step 2 and step 3 alternately until all the vertices are included in union of set 1 and set 2.

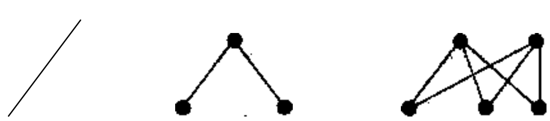
Step 5 If two computed sets following the above steps are distinct, then it is bipartite.

5.2.5 Complete Bipartite Graph:

A bipartite graph *G* is a complete bipartite graph if there is an edge between every pair of vertices taken from two disjoint sets of vertices (one vertex from one set$V_1$and the other from set V_2).

Complete bipartite graph G is denoted by K_{mn}, where *m* and *n* are the number of vertices in two distinct subsetsV_1 and V_2.

Some examples of complete bipartite graphs are shown in Figure.



(a) $K_{1,1}$ (b) $K_{1,2}$ (c) $K_{2,3}$ (d) $K_{2,2}$

Example 5.12

How many edges do the complete bipartite graph, $K_{m\ n}$ have?

Solution:

The vertex set of $K_{m\ n}$ consists of two disjoint sets A and *B*.

A contains *m* vertices and *B* contains *n* vertices.

Each vertex in A is adjacent to each vertex in *B.*

No two vertices either in A or in *B* are adjacent.

Hence, the degree of each vertex in *A* is *n*, and the degree of each vertex in *B* is m.

Therefore, the sum of the degrees is 2 * *m** *n*, and so there are *m** *n* edges (as per the *handshaking* theorem).

Note : Complete bipartite graph $K_{m,n}$ has m + n vertices and m * n edges.

Km, n is regular if m= n.

Example 5.13

Prove that a graph which contains a triangle cannot be bipartite.

Solution

In a bipartite graph, the vertices should be divided into two distinct subsets.

The number of vertices of the given graph is 3, as it is a triangle. So, it is not possible to divide the vertices into two disjoint set of vertices since each edge is joined by the rest two edges.

Hence, this graph may not be a bipartite graph.

5.3 SUBGRAPH

If G and H are two graphs with vertex sets V(G) and V(H) and edge sets E(G) and E(H),respectively, such that V(H)⊆ V(G) and E(H)⊆E(G), then we say that *H* is a subgraph of *G*or G is a super-graph of *H.*

In other words, if *H* is a subgraph of G, then all the vertices and the edges of *H* are in G and each edge of *H* has the same endpoints as in *G.*

Now if V(H)= V(G) *and* E(H) C E(G), then we say that *H* is a ***spanning subgraph*** of G.

A *spanning subgraph* is a subgraph that contains all the vertices of the original graph.

If H is a subgraph of G, then

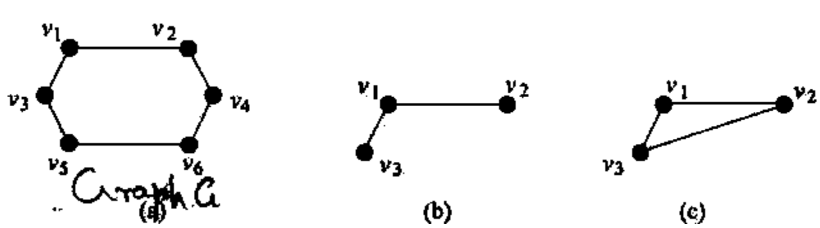
(a) All the vertices of H are in G.

(b) All the edges of H are in G.

(c) Each edge of H has the same endpoints in *H* as in *G.*

For example

A graph G is shown below and its one subgraphis in Figure (b), but the graph shown in Figure (c) is not a subgraph of *G,* as no edge between v_3 and v_2 is present in the original graph G.



Note :

Suppose a graph G has n number of vertices (i.e., |V| = n) and m number of edges (i.e., |E|=n)

Then, number of non-empty *subsets* of V as 2^n - 1 and number of subsets of *E* as 2^m.

Thus, the total number of non-empty subgraphs of *G* is $(2^n - 1) * 2^m$.

Example 5.14

Prove that the number of spanning subgraphs of a graph *G* with m vertices is 2^m

Proof :

Number of spanning subgraphs with 0 (zero) edge and *m* vertices is ^mC_o

Number of spanning subgraphs with 1 edge and *m* vertices is $^mC_1$.

Number of spanning subgraphs with 2 edges and *m* vertices is $^mC_2$.

….. ….

Number of spanning subgraphs with r edges and *m* vertices is ^mC_r.

……. ……..

Number of spanning subgraphs with m edges and *m* vertices is ^mC_m.

Total number of spanning subgraphs

$$= {}^mC_o + {}^mC_1 + {}^mC_2 + \bullet\bullet\bullet + {}^mC_m$$
$$= 2^m \text{ (by binomial theorem)}$$

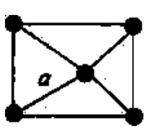
Example 5.15

For the graph G draw the subgraphs

(a) *G* — *e* (here, *e* is one edge)

(b) *G* — *a* (here, *a* is one vertex)

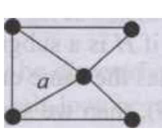
e

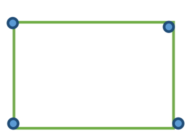


G

Solution :

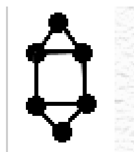
The subgraphs are shown below



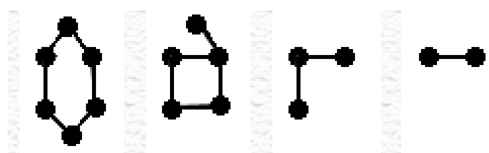


G –e **G – a**

Example 5.16 : Draw some subgraphs of the graph



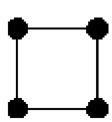
Solution :



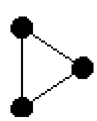
5.4 OPERATIONS ON GRAPH

In this section some operation on graph are discussed.

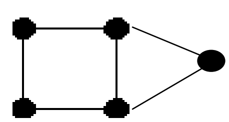
i. **Union of two graphs** G1 and G2 will be another graph G such that $V(G1 \cup G2) = V(G1) \cup V(G2)$ and $E(G1 \cap G2) = E(G1) \cap E(G2)$
If no common vertex is present in between G1 and G2 then the resultant graph will be disconnected.



G1



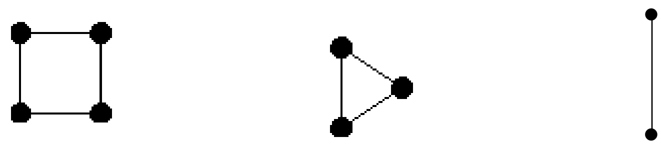
G2



G1 U G2

ii. **Intersection of two graph** G1 and G2 will be another graph G such that

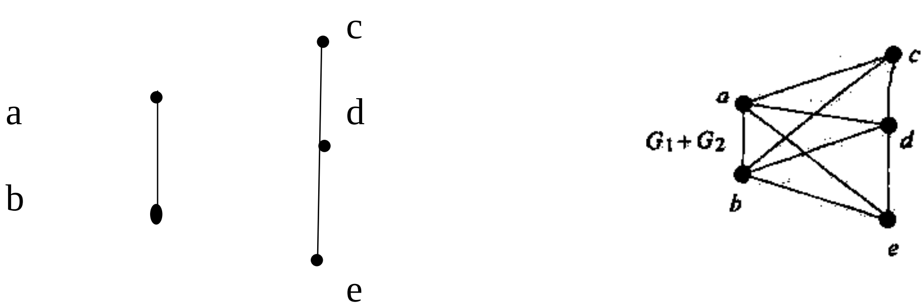
$$V(G_1 \cap G_2) = V(G_1) \cap V(G_2) \neq \Phi and E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$$



G1 G2 G1 ∩G2

iii. **Sum of two graphs**

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_1)$ be two graphs such that $V_1 \cap V_2 \neq \Phi$. The sum of two graphs G_1 and G_2 is G1+G2 is defined as the graph G in which vertex set is V_1+V_2 and the edge set consists of the edges in E_1 and E_2 and the edges joining each vertex of V_1 with each vertex of V_2.



G_1 G_2 $G_1 + G_2$

iv. ***Complement***: The complement *G'* of *G* is defined as a simple graph (parallel edge and self-loop are ignored) with the same vertex set as *G, and where two vertices u* and v are adja*cent only when they are not adjacent in G.*

G G'

v. **Product of two graphs***G_1* and $G_2$is defined as G = *(*V_1 *U* V_2, V_1x

V_2)where $V_1 \cup V_2$ is the union of the vertex sets V_1 of G_1 and $V_2$of G_2, and $V_1 \times V_2$ is the *cross product* to compute the edge set of the resultant graph G.

5.5 REPRESNTATION OF GRAPH

Diagrammatic (graphical) representation of a graph is very *convenient for* visual study, but it is practically feasible only when the number of vertices and edges of the graph is reasonably small. So, we need some other reasonable ways to represent graphs with large number of vertices and edges. These *representations are* also expected to be useful in computer programming. Some representations for undirected as well as directed graphs are discussed below.

5.5.1 Matrix (Adjacency Matrix) Representation

The adjacency matrix is commonly used to represent graphs for computer processing. In such representation, an n x n *Boolean* (1,0) matrix is used where a 1 at position (u, v) indicates that there exists an edge from vertex *u* to v, and a 0 at position(u, v) indicates that there is no edge reachable directly from *u* to v.

If the graph is undirected, then its corresponding adjacency matrix will be symmetric.

(i) Matrix presentation of undirected graph

If an undirected graph G consists of n vertices (assuming that the graph has no parallel edge), then the adjacency matrix of G is an n x n matrix A =$[a_{ij}]$ and defined as follows:

$$a_{i,j} = \begin{cases} 1, if\ there\ is\ an\ undirected\ edge\ between\ vi\ and\ vj \\ 0, \quad if\ there\ is\ no\ edge\ between\ vertices\ vi\ and\ vj \end{cases}$$

Some observations from matrix representation of undirected simple graph:

(a) $a_{i,j} = a_{j,i}$ for all *i* and *j*, i.e., the matrix is symmetric.

(b) Diagonal elements of the matrix are zero (0) (as the simple graph possesses no self loop).

(c) The degree of a vertex is the sum of the 1s in that row.

(d) Let G be a graph with *n* vertices: V_1, V_2, V_3,…,V_n and A be the adjacency matrix of G. Let *B* be the matrix computed as follows:

$$B = A + A^2 + A^3 + \cdots + A^n (n > 1)$$

Now, B is connected if and only if B has no zero entry.

(ii) **Matrix representation of directed graph**

Let G a be a directed graph (digraph) consists of n vertices (assuming that the graph has no parallel edge. The adjacency matrix of G is an n x matrix A $=[a_{i,j}]$ and is defined as follows

$a_{i,j} = \{1,$ if there is a directed edge between vertices v_i and v_j
0, otherwise

Some observations from the matrix representation of directed simple graph.

a. $a_{i,j} \neq a_{j,i}$ for all i and j
b. Diagonal elements of the matrix A are 0
c. The sum of 1 in any column j of A is equal to the in-degree of vertex v_j.
d. The sum of 1 in any row i of A is equal to the out-degree of vertex v_i

5.5.2 Incidence Matrix Representation of Graph

Let G be a graph with n verities and e edges.

The incidence matrix is defined as an nxe matrix $B=[b_{i,j}]$ where

$$b_{i,j} = \begin{cases} 1, & if\ jth\ edge\ ej\ is\ incident\ with\ the\ ith\ vertx \\ 0, & otherwise \end{cases}$$

Example 5.17 :

Write the incidence matrix of the graph G given in figure

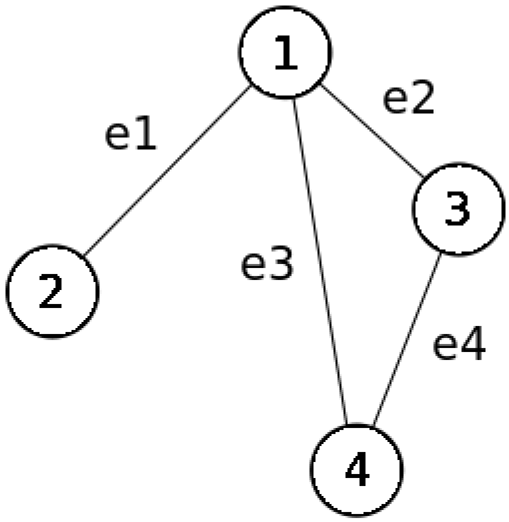


Figure 5. 13

Solution :

The incidence matrix is

Edge / Vertex	e1	e2	e3	e4
Vertex 1	1	1	1	0
2	1	0	0	0
3	0	1	1	1
4	0	0	1	1